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ELEMENTS OF ALGEBRA

FOR BEGINNERS

BY

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PREFACE

It is becoming more and more evident each year that a pupil receives more culture, makes greater progress in study, and, consequently, is better prepared for life, by beginning algebra early, than by spending the same amount of time on arithmetic. And when properly presented by both teacher and text-book, the study of algebra has never failed to be interesting and profitable.

This book is written with the hope that it may contribute largely toward making algebra both valuable and popular to the young. To accomplish this end we have made the *equation* the most prominent idea of each chapter. In the first section we have made the transition from arithmetic to algebra so simple and easy that no pupil will experience any difficulty in mastering the elements of the science.

A third feature of this book is a clear and logical discussion of all those processes that throw light and life into the operations of arithmetic.

We express our sincere thanks for the assistance we have received from experienced teachers.

GEORGE W. HULL.

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ELEMENTS OF ALGEBRA

I. ALGEBRAIC NOTATION

1. Symbols. In Arithmetic the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, called *figures*, are used to represent numbers. In Algebra *the letters of the alphabet,* in addition to the figures of Arithmetic, are employed to represent *numbers* or *quantities.*

Thus, in Arithmetic, if a stone weighs 6 pounds and a brick 4 pounds, they both weigh 6 plus 4, or 10 pounds. In Algebra, if a stone weighs a pounds and a brick b pounds, they both weigh a plus b pounds.

If a represents a number, $2a$ will represent 2 times that number, $3a$ will represent 3 times that number, etc.

2. Addition. The symbol of addition is $+$, read **plus.** The symbol of equality is $=$, read **equals.**

1. What is the sum of 2 times a number plus 3 times that number? Evidently 5 times that number.

2. What is the sum of $2a$ and $3a$? Evidently $5a$.

3. What is the sum of $3a+4a$? Of $5a+7a$?

4. What is the sum of $2a+3a+4a$? Of $4b+5b+6b$?

5. What is the sum of $4n+6n+8n$? Of $2x+3x+6x+7x$?

6. What is the sum of $3n+10n$? What is the value when $n=2$? When $n=3$?

7. What is the value of $4x+3x+2x$ when $x=2$? When $x=5$?

8. What is the value of $4y+3y+5y$ when $y=4$? When $y=10$?

9. What is the value of $6n+4x$ when $n=3$ and $x=4$? When $n=5$ and $x=8$?

10. What is the value of $4m+3y+2m+6y$ when $m=4$ and $y=3$? When $m=5$ and $y=10$?

3. Subtraction. The symbol for subtraction is $-$, **read minus.**

1. What is the difference between 5 times a quantity and 2 times that quantity? Evidently 3 times that quantity.

2. What is the difference between $5a$ and $2a$? Evidently $3a$.

3. What is the difference between $10a$ and $4a$? Between $12y$ and $5y$?

4. What is the difference between $15x$ and $7x$? Between $18n$ and $10n$?

5. What is the difference between $8x+6x$ and $3x$? Between $16n$ and $4n+12n$?

6. What is the value of $20y-8y$ when $y=4$? When $y=12$?

7. What is the value of $2n+8n-5n$ when $n=10$? When $n=25$?

8. What is the value of $12x+20x-15x$ when $x=12$? When $x=20$?

9. What is the value of $8b+12a-6b$ when $b=8$ and $a=10$?

10. What is the value of $18x+4y-10x-4y$ when $x=10$ and $y=8$?

4. Multiplication. The symbol of multiplication is $\times$, read **times**.

The product of two or more quantities expressed by letters is indicated by writing the letters side by side without any sign between them.

Thus, $a \times b$ is written ab, and $a \times b \times c$ is written abc.

1. What is the product of $x \times y$? Of $m \times n$?
2. What is the product of $a \times b \times d$? Of $m \times n \times p$?
3. What is the value of mn when $m = 4$ and $n = 5$?
4. What is the value of abc when $a=3, b=4$, and $c = 5$?
5. What is the value of $2xyz$ when $x = 5$, $y = 6$, and $z = 7$?

The **Coefficient** of a quantity is the number prefixed to it, showing how often it is taken.

Thus, in $4x$, 4 is the coefficient of x.

The **Factors** of a quantity are the quantities which, when multiplied together, will produce it.

Thus, the factors of ab are a and b.

The **Power** of a quantity is the result obtained by using the quantity several times as a factor.

Thus, $a \times a = a^2$, read *a square;*

$a \times a \times a = a^3$, read *a cube;*

$a \times a \times a \times a = a^4$, read *a fourth.*

6. What is the value of 4^2? Of 6^3?
7. What is the value of n^2 when $n = 5$? Of n^3? Of n^4?
8. What is the product of $a \times a \times a \times a \times a \times a$?
9. What is the difference between n^2 and $2n$ when $n = 6$?
10. What is the difference between x^3 and $3x$ when $x = 8$? When $x = \frac{1}{2}$?

The **Exponent** is a number or a letter, written slightly above the quantity on the right, to indicate how often the quantity is used as a factor.

Thus, in a^3, 3 is the exponent.

11. What is the value of x^2y^2 when $x=2$ and $y=3$?

12. What is the value of x^2y^3 when $x=3$ and $y=2$?

13. Which is the greater, and how much, $4x$ or x^4 when $x=3$? When $x=\frac{1}{3}$?

14. What equal quantities added together will give $7x$? What equal quantities multiplied together will produce x^7?

15. When $x=3$, what is the value of $3x$? Of x^3? Of $4x$? Of x^4?

A point ($\cdot$) is sometimes used to denote multiplication.

Thus, $2 \cdot 3 \cdot 4$ denotes $2 \times 3 \times 4$.

5. Division. The symbol of division is $\div$, read **divided by**.

Division may be expressed in several ways.

Thus, $\frac{a}{b}$ or $a : b$ indicates the same as $a \div b$.

1. What is the value of $x \div y$ when $x=40$ and $y=10$?

2. What is the value of $mn \div a$ when $m=8$, $n=10$, and $a=5$?

3. What is the value of $x^3 \div y$ when $x=10$ and $y=25$?

4. What is the value of $a^3 \div b^2$ when $a=8$ and $b=4$?

5. What is the value of $x^3 \div 3x$ when $x=9$? When $x=6$?

6. What is one of the two equal factors of 16? Of 36? Of 64? Of a^2? Of x^2? Of n^2?

7. What is one of the three equal factors of 8? Of 27? Of 64? Of x^3? Of y^3? Of n^3?

The **Root** of a quantity is one of the equal factors of the quantity. The process of finding the root of a quantity is called **Evolution.**

The symbol of evolution is $\sqrt{\ }$, called the **Radical Sign.** The figure placed in the angle of the sign is the **Index.** When no index is used the square root is understood.

Thus, $\sqrt{4}$ indicates the square root of 4.

$\sqrt[3]{x}$ indicates the cube root of x.

$\sqrt[4]{x}$ indicates the fourth root of x.

8. What is the value of $\sqrt{121}$? Of $\sqrt[3]{125}$?

9. What is the value of $\sqrt{x}$ when $x = 100$?

10. What is the value of $\sqrt{ab}$ when $a = 18$ and $b = 2$?

11. What is the value of $\sqrt[3]{mn}$ when $m = 72$ and $n = 3$?

12. What is the value of $\sqrt{x^2}$? Of $\sqrt{x^2y^2}$?

13. What is the value of $\sqrt[3]{n^3}$? Of $\sqrt[3]{x^3y^3z^3}$?

14. Write the sum of x and y; their difference; their product; their quotient; their square; their square root; their cube root.

6. Signs of Aggregation. The symbols ( ), called **Parentheses,** are used to indicate that the quantities inclosed are to be subjected to the same operation.

Thus, $3(a + b)$ indicates that $a + b$ is to be multiplied by 3.

The *Brackets* [ ], the *Braces* { }, and the *Vinculum* ——, are used for the same purpose.

Thus, $3(a + b)$, $3[a + b]$, $3\{a + b\}$, and $3 \times \overline{a + b}$, all mean that $a + b$ is to be taken 3 times.

The **Sign of Continuation** is $\cdots$, and stands for *and so on.*

Thus, $a + a^2 + a^3 \cdots a^n$ is read a plus a^2 plus a^3, *and so on* to a^n.

The **Signs of Inequality** are $>$ and $<$, and are read *is greater than* and *is less than*, respectively.

Thus, $a > b$ is read *a is greater than b*, and $c < d$ is read *c is less than d*.

Read the following expressions:

1. $4(a-b)$.
2. $y(m+n)$.
3. $(a+b)(a-b)$.
4. $a+n(a-b)$.
5. $m-\overline{a+b-c}$.
6. $3(a-\overline{b+c})$.
7. $x+x^2+x^3 \cdots x^m$.
8. $(a+\sqrt{b+c})(m+n)$.
9. $(x+y)^2-(x-y)^2$.
10. $n(\overline{a^2-2ab+b^2})$.
11. $(m^3+n^3) \div (m+n)$.
12. $(a^2+b+c^2)(a^2-b+c^2)$
13. $\{a-(n-\overline{a+b})\}m$.
14. $aaa \cdots$ to n factors.

Find the value of the following expressions when $a=6$, $b=4$, $c=8$, and $x=10$:

15. $(a+b)c$.
16. $(a-b)-2$.
17. $(a+b)(a-b)$.
18. $(a+b+c)x$.
19. $(a^2-b^2)x$.
20. $(c^2-b^2)a$.
21. $\sqrt{a+x}+\sqrt{ax+b}$.
22. $(a^3-b^3) \div (a-b)$.

7. Algebraic Expressions. In the expression $2a+3b-4c$, $2a$, $+3b$, and $-4c$ are called **Terms**. When a term has the *plus* sign prefixed to it, it is called a *positive term;* and when a *minus* sign is prefixed to a term, it is called a *negative term.*

1. Name the terms and tell the kind of each term in $3x-4y+b$; $4-6x^2-y$; $x^2-2xy+y^2$; m^2-n^2.

When an expression consists of one term, it is called a **Monomial**; when it consists of two or more terms, it is called a **Polynomial.** A polynomial of two terms is a **Binomial**, and one of three terms a **Trinomial.**

Thus, $4\,ab$ is a monomial,
$a+b$ is a binomial,
$a+b-c$ is a trinomial,

and $a+b$, $a+b-c$, and $m+n+a-b$ are polynomials.

What is the value of each of the following expressions when $a=4$, and $b=2$?

2. $2\,a+b$.
3. $a+3\,b$.
4. $a^2+2\,ab+b^2$.
5. a^4+a^2+1.
6. $a^2-2\,ab+b^2$.
7. a^6+a^3+1.

When terms have the *same letters* affected by the *same exponents* they are *similar.*

Thus, $4\,ab^2$ and $-3\,ab^2$ are *similar.*

NOTE. Neither the sign nor the coefficient affects the similarity of the terms.

8. Select the similar terms from the following expressions:

$2\,ab^2$, $3\,a^2b$, $4\,a^2b^2$, $-2\,a^2b$, $-2\,ab^2$, $-3\,a^2b^2$, $-ab^2$, $-a^2b^2$, $8\,x^3y$, $8\,xy^3$, $14\,ab^2$, $9\,x^3y$, $-xy^3$, x^3y^3, $-6\,a^2b^2$.

AXIOMS

8. An **Axiom** is a self-evident truth.

1. *If equals are added to equals, the sums will be equal.*

2. *If equals are subtracted from equals, the remainders will be equal.*

3. *If equals are multiplied by equals, the products will be equal.*

4. *If equals are divided by equals, the quotients will be equal.*

5. *Like powers of equal quantities are equal.*

6. *Like roots of equal quantities are equal.*

7. *Things that are equal to the same thing are equal to each other.*

SOLUTION OF PROBLEMS BY ALGEBRA

9. An **Equation** is an expression of the equality between two quantities.

Thus, $2+4=6$, $2x+4=10$ are equations.

1. Find the value of x when $2x=10$. *Ans.* $x=5$.
2. Find the value of y when $8y=24$.
3. Find the value of x when $6x+4x=90$.

Solve the following equations:

4. $10x=100$.
5. $8x-2x=42$.
6. $4x+2x=120$.
7. $10x-3x=70$.
8. $9x+3x=96$.
9. $9y+2y-y=160$.
10. $8y+10y-3y=75$.
11. $10x-2x+12x=100$.

TRANSPOSITION OF TERMS IN EQUATIONS

10. The **First Member** of an equation is the quantity on the left of the sign of equality.

The **Second Member** of an equation is the quantity on the right of the sign of equality.

Transposition is the process of changing a quantity from one member of an equation to the other.

1. Solve the equation $8x-5=6x+7$.

PROCESS

$$\begin{array}{rcl} 8x-5 & = & 6x+7 \\ +5 & = & +5 \\ \hline 8x & = & 6x+7+5 \\ 6x & = & 6x \\ \hline 8x-6x & = & 7+5 \end{array}$$

or $2x=12$

$x=6$, *Ans.*

Adding 5 to both members of the equation, we have $8x=6x+7+5$.

Subtracting $6x$ from both members of this equation, we have $8x-6x=7+5$.

Collecting the terms of this equation, we have $2x=12$, or $x=6$.

By examining this process we see that in changing -5 and $+6x$ from one member of an equation to the other their signs are changed. Hence we have the following

RULE. *A term may be transposed from one member of an equation to the other if at the same time the sign is changed.*

EXAMPLES

Transpose and find the value of x in the following equations:

2. $x+8=10.$
3. $5x-4=6.$
4. $6x-12=24.$
5. $9x-4=32.$
6. $10x+14=44.$
7. $12x-20=40.$
8. $8x-5=35.$
9. $7x+5=40.$
10. $9x-8=64.$
11. $6x=5x+4.$
12. $7x=18-2x.$
13. $5x=12+4x.$
14. $12=4x-8.$
15. $3x+4=2x+6.$
16. $5x-6=3x+4.$
17. $6x-4-2x=20.$

PROBLEMS

11. 1. The sum of two numbers is 60, and the greater is 5 times the less. What are the numbers?

PROCESS

Let $\quad x =$ the smaller number.

Then $\quad 5x =$ the larger number.

And $\quad x+5x=60,$

or $\quad 6x=60,$

$x=10$, the smaller number,

and $\quad 5x=50$, the larger number.

2. The sum of two numbers is 108, and the larger is 8 times the smaller. Find the numbers.

3. A and B have 450 acres of land. How many acres has each, if A has 4 times as many as B?

4. A is 6 times as old as B, and the sum of their ages is 98 years. What is the age of each?

5. M and N have 1818 cattle. How many has each, if M has 8 times as many as N?

6. P and Q have $120,000. What is the fortune of each, if P has $21,000 more than Q?

7. R, S, and T have $150,000. How much has each, if R has $10,000 more than S, and S has $10,000 more than T?

8. A, B, and C have $210,000. How much has each, if A has twice as much as B, and B twice as much as C?

9. The difference between two numbers is 120, and the larger is 5 times the smaller. What are the numbers?

10. A horse cost $160 more than a wagon. What was the cost of each, if the horse cost 5 times as much as the wagon?

11. If 120 is subtracted from 5 times a number, the remainder will equal 3 times the number plus 120. Find the number.

12. If $50 is subtracted from 10 times what A paid for a cow, the remainder will be 5 times the cost of the cow plus $50. What was the cost of the cow?

PROBLEMS INVOLVING LITERAL NOTATION

12. 1. A is n years old. What was his age b years ago? What will be his age c years hence?

2. If w represents a man's daily wages, how much will he earn in x days? In y weeks?

3. If x represents any number, what will represent the next higher number? The next lower number?

4. If x represents the units' digit, and y the tens', what will represent the number?

5. If h represents the hundreds' digit, t the tens' and u the units', what will represent the number? What will represent the number reversed?

6. How many cents in a dollars, b quarters, and c dimes?

7. What is the distance around a rectangular room that is x feet long and y feet wide?

8. Find the surface of a cube that is x feet on a side. Find its contents in cubic yards.

9. How many cords are there in a pile of wood a feet long, b feet wide, and c feet high?

10. If A can do a piece of work in a days, how much of it can he do in one day? How much in n days?

NOTE. If A can do a piece of work in 5 days, in 1 day he can do $\frac{1}{5}$ of it. If he can do the work in a days, in 1 day he can do $\frac{1}{a}$ of it.

11. If A can do a piece of work in x days, and B in y days, how much can they together do in one day? How much in n days?

12. What is the sum of $x + x + x \cdots$ to n terms?

13. What is the product of $xxx \cdots$ to n factors?

14. How many acres of land are there in a rectangular field a rods long and b rods wide?

II. FUNDAMENTAL PROCESSES

ADDITION

13. **Addition** is the process of uniting several quantities into one equivalent expression, called the **Sum**.

14. To add similar terms.

1st. *When the terms have like signs.*

1. Find the sum of $2a$, $3a$, and $6a$.

PROCESS

$$\begin{array}{l} 2a \\ 3a \\ \underline{6a} \\ 11a, \textit{ Ans.} \end{array}$$

Since the quantity a is taken two times in the first term, three times in the second, and six times in the third, it is evidently taken $2 + 3 + 6$, or eleven times. Hence the sum is $11a$. Therefore, to add similar terms with like signs we have the following

RULE. *Add the coefficients, and prefix the sum with its proper sign to the common literal part.*

EXAMPLES

2.	3.	4.	5.	6.
$3x$	$5y$	$2ab$	$-3xy^2$	$-4am^2$
$4x$	$8y$	$4ab$	$-7xy^2$	$-5am^2$
$7x$	$9y$	$6ab$	$-8xy^2$	$-8am^2$
$\underline{8x}$	$\underline{7y}$	$\underline{8ab}$	$\underline{-5xy^2}$	$\underline{-6am}$

Find the sum of:

7. $4x$, x, $18x$, $20x$.

8. $7a$, $8a$, $9a$, $10a$.

9. $8xy$, $4xy$, $6xy$, $9xy$.

10. $-3a$, $-4a$, $-8a$, $-10a$.

11. $-6x^2, \ -8x^2, \ -6x^2, \ -9x^2.$

12. $-a, \ -2a, \ -10a, \ -16a.$

13. $9y, \ 8y, \ 16y, \ 24y.$

14. $11ay, \ 13ay, \ 14ay, \ 16ay.$

15. $3am, \ 6am, \ 12am, \ 24am, \ 48am, \ 20am.$

16. $4mn, \ 10mn, \ 16mn, \ 3mn, \ 9mn, \ 18mn.$

17. $40x^2y^2, \ 20x^2y^2, \ 62x^2y^2, \ 80x^2y^2, \ 11x^2y^2, \ 42x^2y^2.$

18. $9xy^2, \ 10xy^2, \ 11xy^2, \ 12xy^2, \ 13xy^2, \ 14xy^2.$

19. $-6n^2, \ -8n^2, \ -10n^2, \ -41n^2, \ -12n^2, \ -60n^2.$

20. $-4abc, \ -6abc, \ -10abc, \ -20abc, \ -40abc.$

2d. *When the terms have unlike signs.*

21. Find the sum of $13a$ and $-8a$.

PROCESS

$$\begin{array}{rcr} 13a & = & 5a + 8a \\ -8a & & -8a \\ \hline & & 5a, \textit{ Ans.} \end{array}$$

$13a$ equals $5a + 8a$. If $-8a$ is united with $+8a$, the result is nothing. Therefore, $-8a$ added to $13a$ equals $5a$.

22. Find the sum of $+5a$, $-6a$, $+9a$, and $-4a$.

PROCESS

$$\begin{array}{r} +5a \\ -6a \\ +9a \\ -4a \\ \hline 4a \end{array} \qquad \begin{array}{r} +14a \\ -10a \\ \hline 4a, \textit{ Ans.} \end{array}$$

The sum of the positive quantities $+5a$ and $+9a$ is $14a$; and the sum of the negative quantities $-6a$ and $-4a$ is $-10a$. The sum of $+14a$ and $-10a$ is $4a$.

From these examples we derive the following

RULE. *Find the sum of the coefficients of the positive terms and the sum of the coefficients of the negative terms separately, subtract the less from the greater, and give to the difference the sign of the greater. Prefix this difference to the common literal part.*

Add:

23. 16 and -7.

24. -14 and 8.

25. $17a$ and $-6a$.

26. $-18n$ and $8n$.

27. $-16abc$ and $20abc$.

28. $-4x^2y^2$ and $22x^2y^2$.

29. $12mn$ and $-28mn$.

30. $81ab^2$ and $-62ab^2$.

31. $5a-6a+10a-14a-12a+20a$.

32. $6ab+5ab-6ab-18ab-12ab+31ab$.

33. $7n^2-4n^2+12n^2-40n^2+102n^2-63n^2$.

34. $6ax+24ax-15ax+75ax-42ax-18ax$.

35. $9m^2x-20m^2x+m^2x+20m^2x-18m^2x-6m^2x$.

36. $4x^2y^2-8x^2y^2+12x^2y^2-16x^2y^2+5x^2y^2-40x^2y^2$.

15. To add dissimilar terms.

1. Find the sum of $6a$, $5b$, and $-c$.

PROCESS

$$\begin{array}{l} 6a \\ 5b \\ -c \\ \hline 6a+5b-c, \ \textit{Ans.} \end{array}$$

Since the terms are dissimilar, they cannot be united by adding their coefficients; the sum is indicated by connecting them with their proper signs, and we have $6a+5b-c$.

2. Find the sum of $2a-3b$, $5b-6c$, $4a-5c$, $-a+8b$, and $2a-b+c$.

PROCESS

$$\begin{array}{rrr} 2a & -3b & \\ & 5b & -6c \\ 4a & & -5c \\ -a & +8b & \\ 2a & -b & +c \\ \hline 7a & +9b & -10c, \ \textit{Ans.} \end{array}$$

For convenience in adding, we write the quantities so that similar terms are in the same column.

The columns are added as in Art. 14. The sum of the first column is $7a$, of the second column $9b$, and of the third column $-10c$. Hence the sum is $7a+9b-10c$.

From these examples we derive the following

RULE. *Write the quantities so that similar terms shall be in the same column. Add each column separately. Connect the results with their proper sign.*

EXAMPLES

3.	4.	5.
$\begin{array}{r} 4a-5b \\ -6a+8b \\ 13a-7b \\ -5a-2b \\ \hline \end{array}$	$\begin{array}{r} 5ab-\ \ 6xy \\ -8ab+\ \ 9xy \\ 5ab-\ \ 6xy \\ -7ab+12xy \\ \hline \end{array}$	$\begin{array}{rrr} a-5b & & \\ & +6b & -15c \\ 6a-7b & & \\ -a-8b & & +\ \ 5c \\ \hline \end{array}$

6. Add $4a-5b$, $6a+4b$, $a-3b$, $6a+7b$, $a-6b$, $16a-20b$.

7. Add $4x-7y$, $a+7x+4y$, $-6a-8x+9y$, $5a-4y$, $7x-9y$, $18a-16y-11x$.

8. Add $4x^3-5x^2+3x$, $6x^3+2x^2-5x$, $7x^3+9x^2-11x$, $-4x^3-12x^2+15x$, $7x^3+20x^2+6x$.

9. Add $a^3-3a^2b+3ab^2-b^3$, $a^3+3a^2b+3ab^2+b^3$, $4a^3-5a^2b-6ab^2-8b^3$, $-5a^3+3a^2b-7ab^2+b^3$.

10. Add $x^2+2xy+y^2$, $x^2-2xy+y^2$, x^2+xy+y^2, x^2-xy+y^2, $x^2+3xy+y^2$, $x^2-5xy-4y^2$, $-6x^2-6xy-9y^2$, $10x^2-7xy+4y^2$.

11. Add $m^4+4m^3n+6m^2n^2+4mn^3+n^4$, $m^4-4m^3n+6m^2n^2-4mn^3+n^4$.

12. Add $6m^2-5mn+7n^2$, $4m^2+6mn-8n^2$, $5n^2-7m^2+mn$, $4mn-4m^2+4n^2$, $-6m^2-5mn+5n^2$.

13. Add $a^2-2ab+b^2$, $5a^2+7ab+8b^2$, $13a^2-5ab-4b^2$, $-7a^2+6ab-8b^2$, $-a^2-6ab+2b^2$, $-11a^2+b^2$.

14. Show that $x+y+z=b$, if $x=a+b+c$, $y=2a+3b+2c$, and $z=-3a-3b-3c$.

15. Find the value of $x+y+z$, when $x=m^2+2mn+n^2$, $y=2m^2-3mn+2n^2$, and $z=-3m^2+mn-3n^2$.

16. Add $4(a-b)+3(c-d)$, $5(a-b)-(c-d)-8(a-b)+7(c-d)$, $13(a-b)-9(c-d)$.

17. Add $5ax+7(m+n)+3by$, $4ax-3(m+n)+5by$, $-7ax-7(m+n)-9by$, $10by-6ax+14(m+n)$, $9by-11ax-17(m+n)$, $6(m+n)-4ax+5by$.

18. Show that $x+y+z=0$, if $x=m^2+mn+n^2$, $y=m^2-mn+n^2$, and $z=-2m^2-2n^2$.

SUBTRACTION

16. **Subtraction** is the process of finding the difference between two quantities.

The **Minuend** is the quantity from which we subtract.

The **Subtrahend** is the quantity to be subtracted.

The **Remainder** is the result obtained by subtracting.

17. To subtract when the terms are positive.

1. Subtract $5a$ from $9a$.

PROCESS

$$\begin{array}{l} 9a \\ \underline{5a} \\ 4a, \text{ } \textit{Ans.} \end{array}$$

It is evident that 9 times a quantity minus 5 times that quantity equals 4 times that quantity. Hence $9a$ minus $5a$ equals $4a$.

2. Subtract $9a$ from $4a$.

PROCESS

$$\begin{array}{rl} 4a & = 9a - 5a \\ \underline{9a} & \underline{= 9a \qquad\quad} \\ -5a & \qquad -5a, \text{ } \textit{Ans.} \end{array}$$

$4a$ equals $9a-5a$; and $9a$ taken from $9a-5a$ leaves $-5a$. Hence $9a$ subtracted from $4a$ leaves $-5a$.

3. Subtract $m+n$ from a.

PROCESS	
a	If m is taken from a the remainder is $a-m$; but we wish to take m *increased* by n from a; we have therefore taken n less than is required, and the remainder is n too large. Taking n from $a-m$ the true remainder is $a-m-n$.
$m+n$	
$a-m-n$, *Ans.*	

From these examples we see that if we change the signs of the subtrahend, and then add, we shall obtain the same result as given by the solution. Hence the

RULE. *Conceive the signs of the subtrahend to be changed, and proceed as in addition.*

EXAMPLES

Subtract:

4.	5.	6.	7.	8.
$10x$	$5x$	$16ax^2$	$9ax^2$	$12m+4n$
$5x$	$10x$	$9ax^2$	$16ax^2$	$4m+12n$

9. From $3a+b+c$ take $a+2b+2c$.

10. From $12b+8d+7a^2$ take $3b+10d+5a^2$.

11. From $a^2+3ab+b^2$ take $a^2+2ab+b^2$.

12. From $x^4+x^2y^2+y^4$ take $x^4+2x^2y^2+y^4$.

13. From $5x^3+x^2+x+1$ take $3x^3+x^2+x+3$.

14. From $80m^2+60mn+70n^2$ take $79m^2+62mn+69n^2$.

15. From $2a^3+3a^2b+3ab^2+2b^3$ take $a^3+3a^2b+3ab^2+b^3$.

16. From $1+x+x^2+x^3-x^4$ take $x^3+2x^2+x+1+3x^4$.

17. From $a^3+b^3+c^3+3abc$ take $3abc+a^3+b^3+2c^3$.

18. From $4(a+b)+7(c-d)$ take $2(a+b)+13(c-d)$.

18. To subtract when some of the terms are negative.

1. Subtract $-m$ from a.

PROCESS

$$\begin{array}{r} a = a + m - m \\ -m = \qquad\quad -m \\ \hline a + m, \textit{ Ans.} \end{array}$$

a equals $a+m-m$; $-m$ taken from $a+m-m$ leaves $a+m$. Hence $-m$ subtracted from a leaves $a+m$.

2. Subtract $m-n$ from a.

PROCESS

$$\begin{array}{c} a \\ m - n \\ \hline a - m + n, \textit{ Ans.} \end{array}$$

If m is taken from a, the remainder is $a-m$; but we wish to subtract m *diminished* by n from a; we have therefore subtracted n more than is required, and the remainder, $a-m$, is n too small. Adding n to $a-m$ the true remainder is $a-m+n$.

From these examples we again see that if we conceive the signs of the subtrahend changed, and then add, we shall obtain the same result as given in the solution. Hence the

RULE. *Conceive the signs of the subtrahend changed, and proceed as in addition.*

EXAMPLES

	3.	4.	5.	6.	7.	8.
From	a	a	$-a$	$-a$	$14a$	$9a$
Take	c	$-c$	c	$-c$	$-9a$	$-14a$

	9.	10.	11.	12.	13.	14.
From	$-9a$	$-14a$	$-9a$	$-14a$	$14a$	$9a$
Take	$-14a$	$9a$	$14a$	$-9a$	$9a$	$14a$

	15.	16.	17.
From	$a^2+2ab+b^2$	$a-b+c$	$a^2 \qquad +b^2$
Take	$a^2-2ab+b^2$	$a+b-c$	$a^2-2ab+b^2$

18. From $x^2 + 2 + y + y^2$ take $x^2 - xy - y^2$.

19. From $x^2 + y^2$ take $x^2 - 2xy - y^2$.

20. From $a^3 + 3a^2b + 3ab^2 + b^3$ take $a^3 - 3a^2b + 3ab^2 - b^3$.

21. From $x^5 + x^4 + x^3 + x^2 + x + 1$ take $x^5 - x^4 + x^3 - x^2 + x - 1$.

22. From $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ take $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

23. From $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ take $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$.

24. From $3x^m - 6x^ny^m + 5y^n$ take $5x^m - 4x^ny^m - 3y^n$.

25. From $3x^2 + 4y^3 + 7xy + 8z^2$ take $9x^2 - 7y^3 + 7xy - 8z^3$.

26. From $8(a - b) + 7(c + d)$ take $5(a - b) - 8(c + d)$.

27. From $12m^2 - 8mn + 6n^2 - 15$ take $5m^2 + 6mn + 6n^2 - 14$.

USE OF THE PARENTHESES

19. The **Parentheses** are so frequently used in algebra that it is important that the principles which govern their use should be well understood.

PRINCIPLE I. *Parentheses preceded by a plus sign may be removed without changing the signs of the inclosed terms.*

The expression $4a - b + (mx + xy - y^2)$,

indicates that $mx + xy - y^2$ is to be added to $4a - b$. Adding these quantities, we have

$$4a - b + mx + xy - y^2.$$

It is seen that the signs of the terms within the parentheses are *not* changed when the parentheses are removed.

NOTE. After removing the parentheses the terms should be collected when possible.

Remove the parentheses from the following expressions:

1. $x^2+(x^2-xy+y^2)$.
2. $a^2+(ab+b^2-a^2)$.
3. $m^3-n^3+(n^3-m^3)$.
4. $2y^2+(y^2+2xy-3y^2)$.
5. $x^2+y^2+(x^2-4y^2)$.
6. $a^2+2ab+b^2+(2ab-a^2-b^2)$.
7. $m^2+(n^2-mn-m^2)$.
8. $a^2-b^2+(a^2-2ab+b^2)$.

PRINCIPLE II. *A quantity may be inclosed in parentheses if it is preceded by the plus sign, without changing the signs of the inclosed terms.*

This is evident, being the converse of Prin. I.

Express the following as binomials:

9. $x-y+a-b$.
10. $m+n+x-y$.
11. $4-x+y-a$.
12. $a-x+m+n$.
13. $a-b+m+n+x-y$.
14. $c+d+m-n+x+z$.
15. $2-x+a-b+m-n$.
16. $a-b+c-d+m-n$.

PRINCIPLE III. *Parentheses preceded by a minus sign may be removed if the sign of each inclosed term is changed.*

The expression

$$4a-b-(mx+xy-y^2)$$

indicates that $mx+xy-y^2$ is to be subtracted from $4a-b$. Subtracting these quantities, we have

$$\begin{array}{r} 4a-b \\ mx+xy-y^2 \\ \hline 4a-b-mx-xy+y^2. \end{array}$$

It is seen that the signs of the terms within the parentheses are all changed when the parentheses are removed.

Remove the parentheses from the following expressions:

17. $a-(-b)$.

18. $x-(x-y)$.

19. $x-(-x+y)$.

20. $m-(n+m)$.

21. $a+b-(a-b)$.

22. $x-y-(x+y)$.

23. $m+2n-(2m+n)$.

24. $3a-2b-(2a+3b)$.

25. $1-(1-x)+(x-x^2)-(2x-3x^2)$.

26. $3x-2a-(2x-3a+5)+(5-x-a)$.

27. $m-2n^2+(mn+n^2)-(m+mn-n^2)$.

PRINCIPLE IV. *A quantity may be inclosed in parentheses when it is preceded by the minus sign, if the sign of each term inclosed is changed.*

This is evident, being the converse of Prin. III.

Express the following as binomials:

28. $m-n-p+q-d+x+y-z$.

29. $-a+b-m-n+2-3y-2z-m^2$.

30. $x-2y+z-4-m-n-p-q$.

Collect in parentheses the coefficients of x, y, and z in the following expressions:

31. $mx-ny-pz-ax+by-cz$

$$=(m-a)x-(n-b)y-(p+c)z.$$

32. $ax+by-cz-mx-ny-pz$.

33. $2x-3y-4z+ax-by+cz$.

34. $4mx-3ay+5bz-3nx-5cy-8mz$.

Expressions sometimes occur containing two or more signs of aggregation; as,

$$a-\{b-(c-\overline{d+e})\}.$$

Such signs can be removed in succession by beginning with the *inside pair* or with *any pair*.

35. Simplify $a - \{b + (a - \overline{b + c})\}$.

We remove the vinculum first,

$$a - \{b + (a - b - c)\},$$

and then the parentheses,

$$a - \{b + a - b - c\},$$

and finally the braces,

$$a - b - a + b + c.$$

Collecting, $= c$, *Ans.*

Simplify:

36. $a + (b - \overline{c - b})$, $a - (b + \overline{c - d})$.

37. $a - (b - \overline{c - b})$, $m - (n - \overline{b - m})$.

38. $a - \{b - c + (a - b - \overline{b - c})\}$.

39. $x - [4a + b - \{6b - (-8d + 2b - \overline{3b - a})\}]$.

40. $x - [-4a - \{-6b - (-8d + \overline{6b - a + d})\}]$.

MULTIPLICATION

20. **Multiplication** is the process of taking one quantity as many times as there are units in another.

The **Multiplicand** is the quantity taken.

The **Multiplier** is the quantity by which we multiply.

The **Product** is the result of the process.

21. PRINCIPLE I. *The exponent of a letter in the product is equal to the sum of its exponents in the two factors.*

Multiply a^3 by a^2.

$$a^3 = aaa, \text{ and } a^2 = aa.$$

Hence $a^3 \times a^2 = aaa \times aa = aaaaa = a^{3+2} = a^5$.

Also
$$a^4 \times a^3 = a^{4+3} = a^7;$$
$$a^5 \times a^4 = a^{5+4} = a^9;$$
$$a^6 \times a^7 = a^{6+7} = a^{13}.$$

PRINCIPLE II. *The product of two factors having like signs is positive, and having unlike signs is negative.*

Four possible cases may arise:

1st. Multiply $+a$ by $+3$.

$$\begin{array}{l} +a \\ +a \\ \underline{+a} \\ +3a \end{array}$$

This means that $+a$ is to be taken 3 times. Since multiplication is an abbreviated form of addition, and the sum of any number of positive quantities is positive, then $+3 \times +a = +3a$.

2d. Multiply $-a$ by $+3$.

$$\begin{array}{l} -a \\ -a \\ \underline{-a} \\ -3a \end{array}$$

This means that $-a$ is to be taken 3 times. Since the sum of any number of negative quantities is negative, then $+3 \times -a = -3a$.

3d. Multiply $+a$ by -3.

$$\begin{array}{c} +a \\ \underline{3} \\ -(+3a) = -3a \end{array}$$

This means that $+a$ is to be taken 3 times and this result is to be made negative. $3 \times +a = +3a$ by the 1st part, and making it negative, gives $-(+3a)$, or $-3a$.

4th. Multiply $-a$ by -3.

$$\begin{array}{c} -a \\ \underline{3} \\ -(-3a) = +3a \end{array}$$

This means that $-a$ is to be taken 3 times and this result is to be made negative. $3 \times -a = -3a$ by the 2d part, and making it negative, gives $-(-3a)$, or $+3a$.

If a and b stand for *any* two quantities, we have

$$+a \times +b = +ab,$$

$$-a \times +b = -ab,$$

$$+a \times -b = -ab,$$

$$-a \times -b = +ab.$$

This principle briefly stated is: *Like signs give* $+$, *and unlike signs* $-$.

22. To multiply when the multiplier is a monomial.

1. Multiply $4a^2x$ by $2a^3$.

PROCESS

$4a^2x$
$2a^3$
$8a^5x$, *Ans.*

To multiply $4a^2x$ by $2a^3$, we must multiply by 2 and then by a^3.

2 times $4a^2x$ is $8a^2x$,
a^3 times $8a^2x$ is $8a^5x$.

Therefore, $4a^2x$ multiplied by $2a^3$ is $8a^5x$.

2. Multiply $3a^2 - 2ab$ by $3a^2b$.

PROCESS

$3a^2 - 2ab$
$3a^2b$
$9a^4b - 6a^3b^2$, *Ans.*

We multiply each term of $3a^2 - 2ab$ by $3a^2b$, and obtain $9a^4b - 6a^3b^2$.

From these examples we derive the following

RULE. *Multiply each term of the multiplicand by each term of the multiplier.*

To the product of the numerical coefficients annex each literal factor with an exponent equal to the sum of its exponents in both factors.

Make the product positive when the factors have like signs, and negative when they have unlike signs.

EXAMPLES

Multiply at sight:

3.	**4.**	**5.**	**6.**	**7.**	**8.**
-4	5	-6	$-a$	a	$-a$
2	-2	-2	x	$-x$	$-x$
9.	**10.**	**11.**	**12.**	**13.**	**14.**
a^2	x^3	m^4	$-a^5$	x^6	$-m^4$
a^3	x^4	m^2	a^7	$-x^5$	$-m^5$
15.	**16.**	**17.**	**18.**	**19.**	**20.**
x^n	a^{2n}	x^{4n}	x^{m+2}	a^{n-1}	x^{m+3}
x^m	a^n	x^{3n}	x^{m-2}	a^{n+1}	x^{n+2}
21.	**22.**	**23.**	**24.**	**25.**	**26.**
$-a^{n+4}$	$-a^8$	$-n^{4n}$	y^{n+2}	x^{4-n}	a^{2-n}
a^{n-3}	a^{n-6}	$-n^{3n-1}$	y^n	x^{3+n}	a^{n+2}

27. $-a^2 \times -a^3 \times -a^4.$

28. $-a \times -a^2 \times -a^3 \times -a^4.$

29. $-x^3 \times -x^3 \times -x^3 \times -x^3 \times -x^3.$

30. $-y^n \times y^n \times -y^n \times y^n \times -y^n \times -y^n.$

Multiply:

31.	**32.**	**33.**	**34.**
$3x^2 - 27$	$a + b - c$	$2m^2 - n^2$	$x^2 - y^2$
$2x$	$3abc$	$2mn$	$3xy$

35. $a^2 + b^2 - c^2$ by $4abc$.

36. $x^2 + y^2 + xy$ by $3xy$.

37. $x^2 - 2xy + y^2$ by $5xy$.

38. $a^2 - 3ab + b^2$ by $-3ab$.

39. $x^3 + x^2 + x + 1$ by $3x$.

40. $a^4 - 3a^2b^2 + b^4$ by $3ab$.

41. $a^3 - 3a^2b + b^3$ by $-2ab$.

42. $m^2 + mn + n^2$ by $6mn$.

43. $a^3 + a^2 - 1$ by $4a^2$.

44. $a^n - a^2b^2 + b^n$ by $4a^nb^n$.

23. To multiply when the multiplier is a polynomial.

1. Multiply $3a - 2b$ by $2a - 3b$.

PROCESS

$$\begin{array}{l} 3a - 2b \\ \underline{2a - 3b} \\ 6a^2 - 4ab \\ \underline{\quad\ - 9ab + 6b^2} \\ 6a^2 - 13ab + 6b^2, \ \textit{Ans.} \end{array}$$

We multiply $3a - 2b$ by $2a$ and then by $-3b$, and add the products. $2a$ times $3a - 2b$ equals $6a^2 - 4ab$, and $-3b$ times $3a - 2b$ equals $-9ab + 6b^2$.

Adding, we have $6a^2 - 13ab + 6b^2$.

From this example we derive the following

RULE. *Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

EXAMPLES

2.

$$\begin{array}{l} m + n \\ \underline{m + n} \\ m^2 + mn \\ \underline{\quad + mn + n^2} \\ m^2 + 2mn + n^2 \end{array}$$

3.

$$\begin{array}{l} m - n \\ \underline{m - n} \\ m^2 - mn \\ \underline{\quad - mn + n^2} \\ m^2 - 2mn + n^2 \end{array}$$

4.

$$\begin{array}{l} m + n \\ \underline{m - n} \\ m^2 + mn \\ \underline{\quad - mn - n^2} \\ m^2 \qquad\quad - n^2 \end{array}$$

5.

$$\begin{array}{l} a^2 + ab + b^2 \\ \underline{a^2 - ab + b^2} \\ a^4 + a^3b + a^2b^2 \\ \quad - a^3b - a^2b^2 - ab^3 \\ \underline{\qquad\qquad + a^2b^2 + ab^3 + b^4} \\ a^4 \qquad\quad + a^2b^2 \qquad\quad + b^4 \end{array}$$

6.

$$\begin{array}{l} x^2 - x + 1 \\ \underline{x^2 + x + 1} \\ x^4 - x^3 + x^2 \\ \quad x^3 - x^2 + x \\ \underline{\qquad + x^2 - x + 1} \\ x^4 \quad + x^2 \quad + 1 \end{array}$$

Multiply:

7. $x+5$ by $x+5$.

8. $x-5$ by $x-5$.

9. $x+5$ by $x-5$.

10. $x+7$ by $x+3$.

11. $x-7$ by $x-3$.

12. $x-8$ by $x-3$.

13. $x+y$ by $x+y$.

14. $x-y$ by $x-y$.

15. $x+y$ by $x-y$.

16. x^2+y^2 by x^2-y^2.

17. $3x^3+2y^2$ by $2x^2-3y^3$.

18. m^3+n^2 by m^3-n^2.

19. $a^4-a^2b^2+b^4$ by a^2+b^2.

20. m^4+m^2+1 by m^4-n^2+1.

21. a^2-a+1 by a^2+a+1.

22. x^3+x^2+x+1 by $x-1$.

23. $x^4-x^3+x^2-x+1$ by $x+1$.

24. $1+2a+4a^2$ by $1-2a+4a^2$.

25. a^3+y^3 by a^3+y^3.

26. a^n-b^n by a^n-b^n.

27. x^n+y^n by x^n+y^n.

28. a^n-b^n by a^n+b^n.

29. x^n-y^m by x^n+y^m.

30. x^2-b by x^2-c.

31. $a+b+c$ by $a+c$.

32. m^2-mn+n^2 by $m+n$.

33. x^2+xy+y^2 by $x-y$.

34. $a^2-2ab+b^2$ by $a-b$.

35. $x^2+2xy+y^2$ by $x^2-2xy+y^2$.

36. $3a+5b-3c$ by $3a+5b-3c$.

37. $(x^2+y^2)(x+y)(x-y)$.

38. $(m-n)(m+n)(m^2+n^2)(m^4+n^4)$.

39. $(x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)$.

40. $(x-y)(x+y)(x-y)(x+y)(x-y)(x+y)$.

SIMULTANEOUS EQUATIONS

24. If we have the equation $x+y=4$, and

let $x=1$; then $y=3$,

or $x=2$; then $y=2$,

or $x=3$; then $y=1$,

etc. etc.

Since any of these pairs of values will satisfy the equation, it is impossible to determine definitely the values of x and y.

But if we have the two simple equations, $x+y=4$ and $x-y=2$, and are required to find a pair of values which will satisfy both of these equations *at the same time,* we have by adding them,

$$\begin{array}{r} x+y=4 \\ x-y=2 \\ \hline 2x \quad\ =6 \end{array}$$

and $x \quad =3$;

and by subtracting them,

$$\begin{array}{r} x+y=4 \\ x-y=2 \\ \hline 2y=2 \end{array}$$

and $y=1$.

If we substitute these values of x and y in the given equations, we obtain

$$x+y=4 \text{ and } x-y=2,$$
$$3+1=4 \qquad 3-1=2.$$

It is evident that these values of x and y satisfy both equations.

It is also evident that we must have two equations when we use two unknown quantities in order to determine their values.

25. Simultaneous Equations are equations that are satisfied by the *same values* of the unknown quantities.

ELIMINATION

26. Elimination is the process of deducing from two or more simultaneous equations a less number of equations containing a less number of unknown quantities.

27. Elimination by addition or subtraction.

1. Solve the equations $\begin{cases} 2x+3y=16, \\ 2x-3y=4. \end{cases}$

If we add the equations,

$$\begin{array}{lrl} & 2x+3y &= 16 \\ & 2x-3y &= 4 \\ \hline \text{we have} & 4x &= 20 \\ \text{and} & x &= 5. \end{array}$$

If we subtract the equations,

$$\begin{array}{lrl} \text{we have} & 6y &= 12 \\ \text{and} & y &= 2. \end{array}$$

2. Solve the equations $\begin{cases} 2x+4y=14, & (1) \\ 3x+5y=19. & (2) \end{cases}$

Since the unknown quantities do not have the same coefficients in both equations, we must multiply the equations by such numbers as will make the coefficients alike.

$$\begin{array}{lrlr} \text{Multiplying (1) by 3,} & 6x+12y &= 42 & (3) \\ \text{Multiplying (2) by 2,} & 6x+10y &= 38 & (4) \\ \hline \text{Subtracting (4) from (3),} & 2y &= 4 & \\ & y &= 2. & \end{array}$$

Substituting this value of y in (1), $2x + 8 = 14,$

$$2x = 6,$$

$$\therefore \quad x = 3.$$

Ans. $x = 3, \ y = 2.$

Therefore, to eliminate by addition or subtraction, we have the following

RULE. *If necessary, multiply or divide the given equations by such a number as will make the coefficients of one of the unknown quantities equal.*

When the signs of the equal coefficients are unlike, add the equations; when they are alike, subtract them. Solve the resulting equation.

Solve the following equations:

3. $\begin{cases} x - y = 7, \\ x + y = 9. \end{cases}$

4. $\begin{cases} 2x + 3y = 22, \\ 2x - y = 6. \end{cases}$

5. $\begin{cases} 2x - 3y = 4, \\ 5x - 3y = 31. \end{cases}$

6. $\begin{cases} x + 3y = 13, \\ 2x + 5y = 23. \end{cases}$

7. $\begin{cases} 3x + 4y = 32, \\ 2x - y = 14. \end{cases}$

8. $\begin{cases} 11x - 2y = 5, \\ 10x + 2y = 16. \end{cases}$

9. $\begin{cases} 3x + y = 6, \\ 5x + 2y = 11. \end{cases}$

10. $\begin{cases} 9x - 6y = 3, \\ 4x - y = 18. \end{cases}$

11. $\begin{cases} 4x - 3y = 22, \\ 5x + 2y = 39. \end{cases}$

12. $\begin{cases} 6x + 5y = 61, \\ 5x - 6y = 0. \end{cases}$

13. $\begin{cases} 10x - y = 5, \\ 10y - x = 49. \end{cases}$

14. $\begin{cases} 2x - 3y = 17, \\ 4x - 5y = 35. \end{cases}$

15. $\begin{cases} 2x - 3y = 4, \\ x - y = 4. \end{cases}$

16. $\begin{cases} 3x - y = 2, \\ 10x + 2y = 60. \end{cases}$

17. $\begin{cases} 4x + 2y = 22, \\ 6x - 3y = 3. \end{cases}$

18. $\begin{cases} 8x - 2y = 16, \\ 5x + 7y = 43. \end{cases}$

PROBLEMS CONTAINING TWO UNKNOWN QUANTITIES

28. 1. The sum of two numbers is 14, and their difference is 6. What are the numbers?

2. The sum of two numbers is 400, and their difference is 150. What are the numbers?

3. Find two consecutive numbers whose sum is 63.

4. A horse and wagon cost \$300, and the horse cost \$100 more than the wagon. What was the cost of each?

5. A and B together have \$10,000. Two times A's money added to three times B's is \$24,000. How much money has each?

6. If 6 pounds of sugar and 10 pounds of tea cost \$6.30, and at the same price 10 pounds of sugar and 6 pounds of tea cost \$4.10, what is the price of each per pound?

7. A hired 10 men and 8 boys for \$30 a day, and at the same rate 7 men and 12 boys for \$29 a day. How much did each earn in a day?

8. A boy bought 12 oranges and 10 bananas for 56 cents. At another time, at the same price, he bought 5 oranges and 8 bananas for 31 cents. What was the price of each?

9. If 6 horses cost \$300 more than 20 cows, what is the price of each, if a horse and cow cost \$180?

10. Two farms contain 900 acres, the larger contains 300 acres more than the smaller. How many acres are there in each farm?

11. Ten men and eight boys receive \$38 per day for their labor, and 7 men receive \$1 more per day than 20 boys. What are the daily wages of each?

12. Twice the sum of two numbers is 60, and 3 times their difference is 18. What are the numbers?

13. Four yards of cloth and 6 yards of silk are sold for \$ 26, and 3 yards of cloth and 5 yards of silk for \$ 21. What is the price of each per yard?

14. Ten bushels of corn and 12 bushels of wheat cost \$ 14.60. But 12 bushels of corn and 10 bushels of wheat cost \$ 14. What is the price of each?

DIVISION

29. **Division** is the process of finding how many times one quantity is contained in another.

The **Dividend** is the quantity to be divided.

The **Divisor** is the quantity by which we divide.

The **Quotient** is the result of the process.

30. PRINCIPLE I. *The exponent of a factor in the quotient equals its exponent in the dividend minus its exponent in the divisor.*

Divide a^5 by a^3.

$$a^5 = aaaaa, \text{ and } a^3 = aaa.$$

Then $$\frac{a^5}{a^3} = \frac{aaaaa}{aaa} = aa = a^2.$$

Hence $$a^5 \div a^3 = a^{5-3} = a^2.$$

In like manner $$a^{10} \div a^4 = a^{10-4} = a^6.$$

PRINCIPLE II. *The quotient is positive when the dividend and divisor have like signs, and negative when they have unlike signs.*

Thus $+ab \div +b = +a$, since $+a \times +b = +ab$.

$-ab \div +b = -a$, since $-a \times +b = -ab$.

$+ab \div -b = -a$, since $-a \times -b = +ab$.

$-ab \div -b = +a$, since $+a \times -b = -ab$.

31. To divide when the divisor is a monomial.

1. Divide $6\,a^5$ by $2\,a^3$.

PROCESS

$$2\,a^3)\underline{6\,a^5}$$
$$3\,a^2,\ \textit{Ans.}$$

To divide $6\,a^5$ by $2\,a^3$, we must take out the factors 2 and a^3. 6 divided by 2 equals 3, and a^5 divided by a^3 equals a^2. Hence $6\,a^5$ divided by $2\,a^3$ equals $3\,a^2$.

2. Divide $9\,a^5b^3 - 6\,a^4b^5$ by $3\,a^2b^2$.

PROCESS

$$3\,a^2b^2)\underline{9\,a^5b^3 - 6\,a^4b^5}$$
$$3\,a^3b - 2\,a^2b^3,\ \textit{Ans.}$$

Since dividing each term of a quantity divides the entire quantity, we divide each term of $9\,a^5b^3 - 6\,a^4b^5$ by $3\,a^2b^2$, and obtain $3\,a^3b - 2\,a^2b^3$.

From these examples we derive the following

RULE. *Divide each term of the dividend by the divisor. To the quotient of the coefficients annex the literal quantities, giving each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor. Make the quotient + when the dividend and divisor have like signs, and − when they have unlike signs.*

EXAMPLES

3.	4.	5.	6.	7.
$2)\underline{6}$	$2)\underline{-6}$	$-2)\underline{6}$	$-2)\underline{-6}$	$a)\underline{a^2}$
8.	**9.**	**10.**	**11.**	**12.**
$a^2)\underline{a^3}$	$x^6)\underline{x^9}$	$x^4)\underline{x^7}$	$a^8)\underline{a^{14}}$	$n^4)\underline{n^{10}}$
13.	**14.**	**15.**	**16.**	**17.**
$-a^8)\underline{a^{12}}$	$a^6)\underline{-a^{15}}$	$-a^7)\underline{-a^8}$	$-x^7)\underline{x^{10}}$	$-n^3)\underline{-n^8}$
18.	**19.**	**20.**	**21.**	**22.**
$a^n)\underline{a^{3n}}$	$-x^{2n})\underline{x^{4n}}$	$-a^{3n})\underline{-a^{8n}}$	$x^2)\underline{x^n}$	$x^n)\underline{x^3}$

Divide:

23. $12\,a^2x^4$ by $6\,ax$.

24. $9\,a^3x^3$ by $-3\,a^2x$.

25. $-8\,a^4y^4$ by $2\,a^3y^2$.

26. $15\,y^3a^7$ by $-5\,y^3a^2$.

27. $-20\,a^3b^4c$ by $-5\,ab^4c$.

28. $25\,a^2b^{10}$ by $-5\,a^2b^9$.

29. $-16\,a^8b^4$ by $-4\,a^5b^3$.

30. $12\,x^3y^2z$ by $-3\,xy^2z$.

31. $-24\,m^2n^3$ by $6\,mn^2$.

32. a^m by a^n.

33. a^{m+1} by a^{n+1}.

34. $-x^n$ by x^m.

35. $-x^{2n}$ by x^{2m}.

36. $-a^{3n}$ by $-a^{3m}$.

37. $8\,a^2b-4\,ab^2$ by $2\,ab$.

38. $6\,m^2n-15\,m^2n^3$ by $3\,mn$.

39. $42\,x^2y^3-24\,x^3y$ by $6\,x^2$.

40. $7\,x^5-5\,x^4-4\,x^3$ by $-x^2$.

41. $100\,x^2-50\,x$ by $25\,x$.

42. $x^2+2\,xy+y^2x$ by $-x$.

43. $m^2n+mn^2+n^3$ by $-n$.

44. $16\,a^4m^8+10\,a^5m^9$ by $2\,a^4m^7$.

45. $x^6+x^5+x^4+x^3+x^2$ by x^2.

46. $6\,x^{2n}-12\,x^{3n}-18\,x^{4n}$ by $6\,x^n$.

47. $y^{n+2}-y^{n-2}-y^{n-3}$ by y^2.

48. $x^{2n}+x^{2n+1}-x^{3n-2}$ by x^2.

49. $9\,x^2-21\,x^3+27\,x^4$ by $-3\,x^2$.

50. $m^4-m^3-m^2$ by $-m^2$.

51. $25\,x^2y^2-30\,xy^2-45\,x^2y+50\,x^3y^3$ by $5\,x^2y$.

52. $22\,m^3n^2+33\,mn^3-44\,m^2n^2-55\,mn^8$ by $-11\,mn$.

53. $-a^4b^3c^2+a^2b^4c^3-a^3b^2c^4-5\,a^3b^3c^3$ by $-a^2b^2c^2$.

54. $2(a+b)^3-3(a+b)^4-6(a+b)^5$ by $(a+b)^2$.

32. To divide when the divisor is a polynomial.

1. Divide $a^2 + 2ab + b^2$ by $a + b$.

PROCESS

$$\begin{array}{l|l}
a^2 + 2ab + b^2 & a + b, \text{ Divisor.} \\
\cline{2-2}
\underline{a^2 + ab} & a + b, \text{ Quotient.} \\
ab + b^2 & \\
\underline{ab + b^2} &
\end{array}$$

Since the first term of the dividend, a^2, is equal to the product of the first term of the divisor, a, by the first term of the quotient, we can find the first term of the quotient by dividing a^2 by a. a is contained in a^2, a times; a times $a + b$ equals $a^2 + ab$. Subtracting, and bringing down the next term, we have $ab + b^2$.

Since the first term of this new dividend, ab, is equal to the product of the first term of the divisor, a, by the second term of the quotient, we can find the second term of the quotient by dividing ab by a. a is contained in ab, b times; b times $a + b$ equals $ab + b^2$. Subtracting, there is no remainder. Hence the quotient is $a + b$.

From this example we derive the following

RULE. *Arrange both dividend and divisor according to the ascending or descending powers of a common letter.*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the divisor by the first term of the quotient, and subtract the product from the dividend.

Treat the remainder as a new dividend, and proceed as before.

Continue the process until the remainder does not contain the leading letter.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and annex it with the proper sign to the quotient already found.

EXAMPLES

2.

$$\begin{array}{l|l}
a^3 - b^3 & a - b \\
\cline{2-2}
\underline{a^3 - a^2b} & a^2 + ab + b^2 \\
\quad a^2b - b^3 & \\
\quad \underline{a^2b - ab^2} & \\
\qquad ab^2 - b^3 & \\
\qquad \underline{ab^2 - b^3} &
\end{array}$$

3.

$$\begin{array}{l|l}
16x^4 - 8x^2a^2 + a^4 & 4x^2 - a^2 \\
\cline{2-2}
\underline{16x^4 - 4x^2a^2} & 4x^2 - a^2 \\
\quad - 4x^2a^2 + a^4 & \\
\quad \underline{- 4x^2a^2 + a^4} &
\end{array}$$

4.

$$\begin{array}{l|l}
m^4 + m^2n^2 + n^4 & m^2 + mn + n^2 \\
\cline{2-2}
\underline{m^4 + m^3n + m^2n^2} & m^2 - mn + n^2 \\
\quad - m^3n + n^4 & \\
\quad \underline{- m^3n - m^2n^2 - mn^3} & \\
\qquad m^2n^2 + mn^3 + n^4 & \\
\qquad \underline{m^2n^2 + mn^3 + n^4} &
\end{array}$$

Divide:

5. $x^2 - 2xy + y^2$ by $x - y$.

6. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$.

7. $a^3 + a^2b + ab^2 + b^3$ by $a + b$.

8. $a^3 + b^3$ by $a + b$, and $a^5 + b^5$ by $a + b$.

9. $m^3 - n^3$ by $m^2 + mn + n^2$.

10. $x^6 + y^6$ by $x^2 + y^2$.

11. $13x + 15x^3 - 17x^2 - 3$ by $5x^2 - 4x + 3$.

12. $x^6 + 10x - 33$ by $3 + x^2 - 2x$.

13. $27a^3 + b^3$ by $9a^2 - 3ab + b^2$.

14. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.

15. $24x^2 + 28xy + 8xz + 8y^2 + 4yz$ by $8x + 4y$.

16. $a^{10} + b^{10}$ by $a^2 + b^2$.

17. $a^{12} + b^{12}$ by $a^4 + b^4$.

18. $m^4 + m^5 - 2m - 2m^2$ by $m^2 + m$.

19. $x^6 + x^4 - x^3 + x^2 + 1$ by $x^2 + x + 1$.

20. $15m^4 - 19m^2n + 5m^2 + 6n^2 - 3n$ by $5m^2 - 3n$.

21. $y^6 + x^6$ by $y^4 - y^2x^2 + x^4$.

22. $x^3 - 8$ by $x - 2$, and $y^6 + 27$ by $y^2 + 3$.

23. $x^{2n} - 4$ by $x^n - 2$, and $x^6 - 49$ by $x^3 + 7$.

24. $x^2 + x^3 + 7x^4 + 5x^5$ by $x - x^2$.

25. $12x^3 - 17x^2y + 26xy^2 + 15y^3$ by $4x + 3y$.

SPECIAL METHOD FOR DIVISION OF BINOMIALS

33. The following method of division is so important in abridging algebraic work that the pupil should learn to perform it with rapidity and accuracy.

ILLUSTRATIONS

By performing the division we have

$$
\begin{array}{l}
a + b\,)\,a^5 + b^5\,(\,a^4 - a^3b + a^2b^2 - ab^3 + b^4 \\
\qquad\quad \underline{a^5 + a^4b} \\
\qquad\qquad - a^4b + b^5 \\
\qquad\qquad \underline{- a^4b - a^3b^2} \\
\qquad\qquad\qquad a^3b^2 + b^5 \\
\qquad\qquad\qquad \underline{a^3b^2 + a^2b^3} \\
\qquad\qquad\qquad\quad - a^2b^3 + b^5 \\
\qquad\qquad\qquad\quad \underline{- a^2b^3 - ab^4} \\
\qquad\qquad\qquad\qquad\quad ab^4 + b^5 \\
\qquad\qquad\qquad\qquad\quad \underline{ab^4 + b^5}
\end{array}
$$

$$
\begin{array}{r|l}
a-b\,) & a^5-b^5\,(\,a^4+a^3b+a^2b^2+ab^3+b^4 \\
& \underline{a^5-a^4b} \\
& \quad a^4b-b^5 \\
& \quad \underline{a^4b-a^3b^2} \\
& \qquad a^3b^2-b^5 \\
& \qquad \underline{a^3b^2-a^2b^3} \\
& \qquad\quad a^2b^3-b^5 \\
& \qquad\quad \underline{a^2b^3-ab^4} \\
& \qquad\qquad ab^4-b^5 \\
& \qquad\qquad \underline{ab^4-b^5}
\end{array}
$$

By carefully inspecting these two examples we derive the following conclusions:

I. *When the divisor is $a+b$ the signs of the quotient are alternately plus and minus.*

II. *When the divisor is $a-b$ the signs of the quotient are all plus.*

III. *The first term of the quotient is found by dividing the first term of the dividend by the first term of the divisor.*

IV. *Any succeeding term of the quotient may be found by dividing the term of the quotient last found by the first term of the divisor, and multiplying the result by the second term of the divisor.*

Thus, $$\frac{a^3+b^3}{a+b}=a^2-ab+b^2.$$

Since the divisor is $a+b$, the signs of the quotient are alternately plus and minus.

The first term of the quotient is $a^3 \div a = a^2$.

The second term is $(a^2 \div a)b = ab$.

The third term is $(ab \div a)b = b^2$.

Divide the following binomials:

1. $\dfrac{a^2-b^2}{a-b}$.
2. $\dfrac{a^2-b^2}{a+b}$.
3. $\dfrac{a^3-b^3}{a-b}$.
4. $\dfrac{a^4-b^4}{a-b}$.
5. $\dfrac{a^4-b^4}{a+b}$.
6. $\dfrac{a^6-b^6}{a-b}$.
7. $\dfrac{a^6-b^6}{a+b}$.
8. $\dfrac{a^7-b^7}{a-b}$.
9. $\dfrac{a^7+b^7}{a+b}$.
10. $\dfrac{a^8-b^8}{a-b}$.
11. $\dfrac{a^8-b^8}{a+b}$.
12. $\dfrac{a^9-b^9}{a-b}$.
13. $\dfrac{x^9+y^9}{x+y}$.
14. $\dfrac{x^6+y^6}{x^2+y^2}$.
15. $\dfrac{x^9+b^9}{x^3+b^3}$.
16. $\dfrac{x^{10}-y^{10}}{x-y}$.
17. $\dfrac{x^{10}+y^{10}}{x^2+y^2}$.
18. $\dfrac{x^{12}+y^{12}}{x^4+y^4}$.
19. $\dfrac{x^{14}+y^{14}}{x^2+y^2}$.
20. $\dfrac{x^6+y^9}{x^2+y^3}$.
21. $\dfrac{x^{12}+y^6}{x^4+y^2}$.
22. $\dfrac{a^{15}+b^5}{a^3+b}$.
23. $\dfrac{m^{30}+n^5}{m^6+n}$.
24. $\dfrac{x^9-1}{x-1}$.
25. $\dfrac{x^3+27}{x+3}$.
26. $\dfrac{a^3-64}{a-4}$.
27. $\dfrac{1-n^9}{1-n}$.
28. $\dfrac{a^{20}+b^5}{a^4+b}$.
29. $\dfrac{x^6-1}{x-1}$.
30. $\dfrac{m^3-125}{m-5}$.

REVIEW

34. 1. Add $4a-5b+6c$, $7a+8b-5c$, $5b-6c-10a$, $40c-8b+5a$, and $11c+5b-13a$.

2. Add $a^2+2ab+b^2$, $a^2-2ab+b^2$, $3a^2+3ab+3b^2$, $5a^2-5ab+5b^2$, $10a^2-20ab+10b^2$, $11ab-6a^2-6b^2$, $10b^2-14ab+9a^2$, and $20a^2+20ab+20b^2$.

3. Add am^2-bn^3-14mn, $-cm^2+dn^2+12mn$, bm^2+an^2+9mn, and $(c-b-a)m^2-(d-b+a)n^3+mn$.

4. Subtract $8m^2 - 6mn - 5n^2$ from $12m^2 + 10mn - 6n^2$.

5. Subtract $12a^3 - 15a^2b + 15ab^2 - 12b^3$ from $11a^3 + 11a^2b + 11ab^2 + 11b^3$.

6. Subtract $m^5 - 4m^4 + m^2 - 6m + 7$ from $9 - 12m + 2m^2 - 3m^3 + 4m^4 - 5m^5$.

7. Multiply $m^2 + n^2$ by $m^2 + n^2$.

8. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

9. Multiply $m - a + b$ by $m + a + b$.

10. Divide $4x^4 + y^4 - 5x^2y^2$ by $2x^2 + y^2 + 3xy$.

11. Divide $m^5 - 6m^3 - 7m^2 - 6m$ by $m^3 + m^2 + m$.

12. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

13. Find the value of $a - (a - b) + b - \{2b - (-b - \overline{a + c})\}$.

14. Find the value of $-\{a + b - [a + b - (a + b - \overline{a + b})]\}$.

15. Find the value of $\{(a - \overline{b + c}) - b - (a - \overline{c + 1}) - (\overline{b - c} - a)\}$.

16. Divide $a^{10} + y^{15}$ by $a^2 + y^3$.

17. Divide $a^{24} + b^{27}$ by $a^8 + b^9$.

18. Divide $x^{15} - y^{10}$ by $x^3 - y^2$.

19. Solve $\begin{cases} 11x + 8y = -3, \\ 9y - 4x = 13. \end{cases}$

20. Solve $\begin{cases} 8x - 7y = 0, \\ 4x + 3y = 52. \end{cases}$

21. Solve $\begin{cases} 5x + 4y = -17, \\ 6x - 3y = -36. \end{cases}$

III. COMPOSITION AND FACTORING

COMPOSITION

35. Composition is the process of forming *composite* quantities.

36. A **Composite Quantity** is the product of two or more quantities.

PRINCIPLE I. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

Thus, by multiplication,

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{\quad + ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

Also, $(m + n)^2 = m^2 + 2mn + n^2$.

And, $(2a + 3b)^2 = 4a^2 + 12ab + 9b^2$.

EXAMPLES

Write by inspection the values of the following:

1. $(a + x)^2$.
3. $(x + y)^2$.
5. $(b + c)^2$.
2. $(a + n)^2$.
4. $(y + b)^2$.
6. $(m + n)^2$.

7. $(l+n)^2$.
8. $(a+m)^2$.
9. $(r+t)^2$.
10. $(x+2)^2$.
11. $(x+3)^2$.
12. $(y+4)^2$.
13. $(a+5)^2$.
14. $(2+x)^2$.
15. $(7+m)^2$.
16. $(5+n)^2$.
17. $(6+x)^2$.
18. $(2a+3b)^2$.
19. $(5a+4b)^2$.
20. $(3x+3y)^2$.
21. $(2a+5x)^2$.

PRINCIPLE II. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

Thus, by multiplication,

$$\begin{array}{l} a \quad -b \\ \underline{a \quad -b} \\ a^2- \quad ab \\ \underline{\quad - \quad ab+b^2} \\ a^2-2ab+b^2 \end{array}$$

Also, $(m-n)^2=m^2-2mn+n^2$.

And, $(2a-3b)^2=4a^2-12ab+9b^2$.

EXAMPLES

Write by inspection the values of the following:

1. $(a-b)^2$.
2. $(a-x)^2$.
3. $(x-y)^2$.
4. $(y-b)^2$.
5. $(n-m)^2$.
6. $(b-c)^2$.
7. $(l-n)^2$.
8. $(a-m)^2$.
9. $(s-t)^2$.
10. $(x-2)^2$.
11. $(y-3)^2$.
12. $(a-4)^2$.
13. $(2-x)^2$.
14. $(3-m)^2$.
15. $(4-x)^2$.
16. $(2a-2b)^2$.
17. $(3x-3y)^2$.
18. $(4x-5y)^2$.
19. $(2x-5y)^2$.
20. $(7x-3y)^2$.
21. $(3m-7n)^2$.

PRINCIPLE III. *The product of the sum and difference of two quantities equals the difference of their squares.*

Thus, by multiplication,

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \quad - ab - b^2 \\ \hline a^2 \qquad - b^2 \end{array}$$

Also, $$(m + n)(m - n) = m^2 - n^2.$$

And, $$(2a + 3b)(2a - 3b) = 4a^2 - 9b^2.$$

EXAMPLES

Write by inspection the values of the following:

1. $(a + c)(a - c)$.
2. $(m - a)(m + a)$.
3. $(x + n)(x - n)$.
4. $(x + y)(x - y)$.
5. $(a - x)(a + x)$.
6. $(2a + b)(2a - b)$.
7. $(a + 3b)(a - 3b)$.
8. $(4a + 2b)(4a - 2b)$.
9. $(3x - 5y)(3x + 5y)$.
10. $(5x + 5y)(5x - 5y)$.
11. $(3m - 10n)(3m + 10n)$.
12. $(A + 2B)(A - 2B)$.
13. $(y - 3z)(y + 3z)$.
14. $(5x + 4y)(5x - 4y)$.
15. $(ax - by)(ax + by)$.
16. $(ab + 2c)(ab - 2c)$.
17. $(2ab + 3c)(2ab - 3c)$.
18. $(4mn - 3xy)(4mn + 3xy)$.

PRINCIPLE IV. *The product of two binomials having a common term is equal to the square of the common term, the algebraic sum of the unlike terms multiplied by the common term, and the algebraic product of the unlike terms.*

Thus, by multiplication,

$$\begin{array}{l} a + b \\ a + c \\ \hline a^2 + ba \\ \quad + ca + bc \\ \hline a^2 + (b + c)a + bc \end{array}$$

Also, $(a + 5)(a - 3) = a^2 + 2a - 15.$

And, $(a + 5)(a - 7) = a^2 - 2a - 35.$

EXAMPLES

Write by inspection the values of the following:

1. $(m + 2)(m + 3)$.
2. $(x + 4)(x + 5)$.
3. $(y + 3)(y + 4)$.
4. $(a - 2)(a - 3)$.
5. $(n - 3)(n - 7)$.
6. $(c - 5)(c - 10)$.
7. $(x - 5)(x + 8)$.
8. $(x - 2)(x + 10)$.
9. $(y + 7)(y - 3)$.
10. $(m - 8)(m + 3)$.
11. $(x + 10)(x - 12)$.
12. $(y - 9)(y - 4)$.
13. $(a + x)(a + y)$.
14. $(c + d)(c + f)$.
15. $(x + y)(x + z)$.
16. $(a - b)(a + c)$.
17. $(m + x)(m - y)$.
18. $(a - c)(a + b)$.

FACTORING

37. Factoring is the process of resolving a composite quantity into its prime factors.

The factors of a composite quantity are the quantities which, when multiplied together, produce it.

A **Prime Quantity** is a quantity that cannot be produced by the multiplication of other quantities; as

$$17,\ a,\ a + b, \text{ or } a^2 + b^2.$$

By Art. 5, one of the two equal factors of a quantity is its *square root*.

Thus, the square root of $9\,a^2$ is $3\,a$.

I. TO FACTOR MONOMIALS

38. The prime factors of monomials may be obtained by inspection.

Thus, the prime factors of $6\,a^3b^2 = 2 \times 3\,aaabb$.

Factor:

1. $20\,a^2b^2$.	5. $81\,a^3b^3$.	9. $216\,m^2a^2n^2$.
2. $16\,m^3n^3$.	6. $108\,m^6n^3$.	10. $125\,a^3b^3$.
3. $54\,x^5y^4$.	7. $120\,x^4y$	11. $105\,x^4y^6$.
4. $64a^2b^3c^4$.	8. $144\,a^2b^4c^3$.	12. $169\,m^6a^8$.

II. TO FACTOR BINOMIALS

39. When the binomial is the difference of two squares.

1. Factor $a^2 - b^2$.

Since $(a + b)(a - b) = a^2 - b^2$, the factors of $a^2 - b^2$ are $(a + b)(a - b)$.

From this example we derive the following

RULE. *Take the square root of each term.*

The sum of these roots is one factor.

The difference of these roots is the other factor.

Factor:

2. $m^2 - n^2$.	7. $4m^2 - 9n^2$.	12. $m^4 - n^4$.
3. $c^2 - d^2$.	8. $16a^2 - 25b^2$.	13. $m^8 - n^8$.
4. $x^2 - y^2$.	9. $4x^2 - 49y^2$.	14. $m^{16} - n^{16}$.
5. $x^2 - 1$.	10. $36b^2 - 81m^2$.	15. $a^8 - b^4$.
6. $1 - m^2$.	11. $x^2y^2 - a^2b^2$.	16. $a^{2n} - b^{2n}$.

The same principle may be applied to the following binomial forms:

17. Factor $a^2 - (b + c)^2$.
The square root of the first term is a.
The square root of the second term is $b + c$.
Their sum is $a + b + c$.
Their difference is $a - b - c$.
Hence, the factors are $(a + b + c)$ and $(a - b - c)$.

18. Factor $(a + b)^2 - (c + d)^2$.
The first factor is $(a + b + c + d)$.
The second factor is $(a + b - c - d)$.

19. $(a + b)^2 - c^2$.	27. $x^2 - (a - y)^2$.
20. $(x + y)^2 - z^2$.	28. $(x + y)^2 - (m + n)^2$.
21. $(m + n)^2 - p^2$.	29. $(a + b)^2 - (c + d)^2$.
22. $(a - x)^2 - y^2$.	30. $(x - y)^2 - (z - v)^2$.
23. $(l - n)^2 - m^2$.	31. $(m - n)^2 - (n + p)^2$.
24. $x^2 - (y + z)^2$.	32. $(x + 1)^2 - (y - 2)^2$.
25. $a^2 - (b - c)^2$.	33. $(ax + by)^2 - (cz - dv)^2$.
26. $c^2 - (m + n)^2$.	34. $(m^2 + n^2)^2 - (a^2 - b^2)^2$.

Polynomials can sometimes be arranged in the form of the difference of two squares, and factored like the preceding examples.

35. Factor $a^2 + 2ab + b^2 - c^2$.

This may be arranged and factored as follows:

$$a^2 + 2ab + b^2 - c^2 = (a+b)^2 - c^2 = (a+b+c)(a+b-c).$$

36. $x^2 + 2xy + y^2 - z^2$.

37. $x^2 - 2ax + a^2 - c^2$.

38. $y^2 - x^2 - 2xz - z^2$.

39. $m^2 - 2mn + n^2 - p^2$.

40. $a^2 - y^2 + b^2 - 2ab$.

41. $m^2 - n^2 - p^2 + 2pn$.

42. $a^2 - 2xy - x^2 - y^2$.

43. $x^2 - 1 + y^2 + 2xy$.

40. When the binomial is the sum or difference of any powers of two quantities.

THEOREM I. *The difference of any powers of two quantities is divisible by the* DIFFERENCE *of the* SAME *roots of those powers.*

1. Factor $a^3 - b^3$.

The $\sqrt[3]{a^3} = a$, and $\sqrt[3]{b^3} = b$.

Then $\dfrac{a^3 - b^3}{a - b} = a^2 + ab + b^2$, and the factors of $a^3 - b^3$ are $a - b$ and $a^2 + ab + b^2$.

2. Factor $a^3 - b^9$ and $a^6 - b^{15}$.

$$a^3 - b^9 = (a - b^3)(a^2 + ab^3 + b^6).$$

$$a^6 - b^{15} = (a^2 - b^5)(a^4 + a^2b^5 + b^{10}).$$

3. $a^3 - x^3$.

4. $a^5 - b^5$.

5. $a^7 - b^7$.

6. $a^6 - b^9$.

7. $x^3 - y^3$.

8. $x^9 - y^3$.

9. $x^5 - y^{10}$.

10. $m^{10} - y^{15}$.

11. $x^9 - y^{12}$.

12. $m^3 - 125$.

13. $n^2 - 36$.

14. $a^3 - b^{18}$.

15. $1 - b^3$.

16. $8 - b^6$.

17. $a^{10} - b^{15}$.

THEOREM II. *The sum of any powers of two quantities is divisible by the* SUM *of the same* ODD *roots of those powers.*

1. Factor $a^3 + b^3$.

The $\sqrt[3]{a^3} = a$, and $\sqrt[3]{b^3} = b$.

Then $\dfrac{a^3 + b^3}{a + b} = a^2 - ab + b^2$, and the factors of $a^3 + b^3$ are $a + b$ and $a^2 - ab + b^2$.

2. Factor $a^6 + b^6$ and $a^5 + b^{15}$.

$$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4).$$

$$a^3 + b^{15} = (a + b^5)(a^2 - ab^5 + b^{10}).$$

3. $m^3 + n^3$.	**8.** $x^{12} + y^{12}$.	**13.** $a^6 + b^{15}$.
4. $a^5 + b^5$.	**9.** $a^{14} + y^{14}$.	**14.** $n^5 + 1$.
5. $m^7 + b^7$.	**10.** $x^{10} + y^5$.	**15.** $a^5 + b^{20}$.
6. $x^6 + b^6$.	**11.** $m^6 + n^9$.	**16.** $m^6 + n^{12}$.
7. $n^{10} + m^{10}$.	**12.** $x^3 + 27$.	**17.** $y^6 + z^{15}$.

III. TO FACTOR TRINOMIALS

41. When the trinomial is of the second degree.

A trinomial of the second degree will assume one of the six following forms:

FIRST FORM. **1.** Factor $a^2 + 2ab + b^2$.

By Art. 36, Prin. I, this is evidently the square of $a + b$. Hence $a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2$.

Factor:

2. $m^2 + 2\,mn + n^2$.
3. $x^2 + 2\,xy + y^2$.
4. $a^2 + 2\,ax + x^2$.
5. $p^2 + 2\,pm + m^2$.
6. $a^2 + 2\,am + m^2$.
7. $4\,a^2 + 4\,ab + b^2$.
8. $a^2 + 6\,ab + 9\,b^2$.
9. $4\,a^2 + 12\,ab + 9\,b^2$.
10. $16\,x^2 + 24\,xy + 9\,y^2$.
11. $m^2 + 16\,m + 64$.

SECOND FORM. **12.** Factor $a^2 - 2\,ab + b^2$.

By Art. 36, Prin. II, this is evidently the square of $a - b$. Hence,

$$a^2 - 2\,ab + b^2 = (a - b)(a - b) = (a - b)^2.$$

Factor:

13. $m^2 - 2\,mn + n^2$.
14. $c^2 - 2\,cd + d^2$.
15. $y^2 - 2\,yx + x^2$.
16. $a^2 + d^2 - 2\,ad$.
17. $p^2 + q^2 - 2\,pq$.
18. $9\,x^2 - 6\,xy + y^2$.
19. $16\,a^2 - 8\,a + 1$.
20. $25\,a^2 + 16\,b^2 - 40\,ab$.
21. $100\,x^2 + 64\,b^2 - 36\,xb$.
22. $a^2 - 26\,a + 169$.

THIRD FORM. **23.** Factor $a^2 + 7\,a + 12$.

By Art. 36, Prin. IV, the first term of each binomial is a, the product of the other two terms is 12, and their sum is 7. The factors of 12 are

$$1 \times 12,$$
$$2 \times 6,$$
$$3 \times 4.$$

Since the sum of the two numbers is 7, they must be 3 and 4. Hence,

$$a^2 + 7\,a + 12 = (a + 3)(a + 4).$$

Factor:

24. $a^2 + 8a + 12.$

25. $a^2 + 13a + 12.$

26. $x^2 + 9x + 20.$

27. $m^2 + 12m + 20.$

28. $x^2 + 21x + 20.$

29. $m^2 + 10m + 24.$

30. $x^2 + 11x + 24.$

31. $y^2 + 14y + 24.$

32. $z^2 + 25z + 24.$

33. $n^2 + 43n + 82.$

Fourth Form. **34.** Factor $a^2 - 7a + 12.$

This form is factored like the Third Form, with the second term in each factor *negative*. Hence,

$$a^2 - 7a + 12 = (a - 3)(a - 4).$$

Factor:

35. $x^2 - 9x + 14.$

36. $a^2 - 5x + 6.$

37. $m^2 - 13x + 22.$

38. $y^2 - 11x + 30.$

39. $z^2 - 15x + 50.$

40. $m^2 - 20m + 91.$

41. $x^2 - 22x + 105.$

42. $y^2 - 23x + 120.$

43. $a^2 - 18x + 77.$

44. $z^2 - 22x + 85.$

Fifth Form. **45.** Factor $a^2 + 7a - 18.$

This form is factored like the Third Form, with the second term in one factor *positive*, and in the other term *negative*, the positive term being the greater. Hence,

$$a^2 + 7a - 18 = (a + 9)(a - 2).$$

Factor:

46. $x^2 + 6x - 16.$

47. $y^2 + 6y - 27.$

48. $z^2 + 6z - 40.$

49. $m^2 + 6m - 55.$

50. $n^2 + 6n - 72.$

51. $x^2 + 7x - 44.$

52. $y^2 + 9y - 36.$

53. $z^2 + 8z - 65.$

54. $m^2 + 13m - 48.$

55. $n^2 + 20n - 96.$

SIXTH FORM. **56.** Factor $a^2 - 6a - 16$.

This form is factored like the third form, with the second term in one factor *positive* and in the other *negative*, the negative term being the greater. Hence,

$$a^2 - 6a - 16 = (a - 8)(a + 2).$$

Factor:

57. $x^2 - 8x - 48$.

58. $y^2 - 4y - 60$.

59. $m^2 - 5m - 84$.

60. $n^2 - 16n - 80$.

61. $z^2 - 6z - 135$.

62. $s^2 - 2s - 120$.

63. $l^2 - 22l - 240$.

64. $x^2 - 4x - 140$.

65. $y^2 - y - 90$.

66. $z^2 - 7z - 78$.

IV. TO FACTOR POLYNOMIALS

42. When the polynomial contains a common factor.

1. Factor $5a^2c - 10abc + 15ac^2 - 20ac^3$.

By examining the polynomial we see that $5ac$ is a factor common to all the terms. Dividing by $5ac$, we obtain the other factor. Hence,

$$5ac \overline{)5a^2c - 10abc + 15ac^2 - 20ac^3}.$$
$$a \quad - \quad 2b \quad + \quad 3c \quad - \quad 4c^2$$

EXAMPLES

Factor:

2. $a^3 + 2a^2b + ab^2$.

3. $a^3 + 2a^2b + ab^2 - ac^2$.

4. $x^4 + 2x^3 + x^2$.

5. $y^4 + 3y^3 + 3y^2 + y$.

6. $x^3 + 7x^2 + 12x$.

7. $y^3 - 12y^2 - 64y$.

8. $x^4 - xy^3$.

9. $y^6 + yx^5$.

10. $a^9 - ab^8$.

11. $m^{13} - mn^{12}$.

MISCELLANEOUS EXAMPLES

Factor:

1. $x^4 - y^4$.
2. $2x^2 - 4x + 2$.
3. $m^2 + 18m + 81$.
4. $a^2 - (b - c)^2$.
5. $m^6 + n^6$.
6. $x^{10} + y^{10}$.
7. $36y^2 + 12y + 1$.
8. $x^{16} - y^{16}$.
9. $y^2 - 3y - 130$.
10. $a^2 + 2a - 99$.
11. $x^2 - (a + b)^2$.
12. $n - n^7$.
13. $m^2 - 7m - 120$.
14. $(x + y)^2 - (x - y)^2$.
15. $m^2 - 32m + 256$.
16. $(m - n)^2 - p^2$.
17. $(a^2 - b^2)^2 - (c^2 - d^2)^2$.
18. $a^2 - a - 90$.
19. $x^2 + x - 132$.
20. $(x + y)^2 + 2(x + y) + 1$.
21. $m^{12} + m^5$.
22. $z^4 + 20z^2 + 100$.
23. $x^3 - x^2 - x + 1$.
24. $x^3 + x^2 + x + 1$.

HIGHEST COMMON DIVISOR

43. A **Divisor** of a quantity is a quantity that will exactly divide it.

Thus, $3a$ is a divisor of $6a^2$.

44. A **Common Divisor** of two or more quantities is a quantity that will exactly divide each of them.

Thus, $3a$ is a common divisor of $6a^2$ and $9a^2b$.

45. The **Highest Common Divisor** (H. C. D.) of two or more quantities is a divisor of the *highest degree* that will exactly divide each of them.

Thus, $3a^2$ is the H. C. D. of $6a^2$ and $9a^2b$.

PRINCIPLE. *The highest common divisor of two or more quantities is the product of all their common prime factors.*

46. To find the highest common divisor of quantities that can be readily factored.

1. Find the H. C. D. of $6\,a^2b^2x$ and $9\,a^2bx^2$.

PROCESS

$$\begin{array}{r}6\,a^2b^2x = 2 \times 3 \times aa \times bb \times x \\ 9\,a^2bx^2 = 3 \times 3 \times aa \times b \times xx \\ \hline \text{H. C. D.} = 3 \times aa \times b \times x = 3\,a^2bx, \ \textit{Ans.}\end{array}$$

Since the highest common divisor is the product of all the common prime factors (Art. 45, Prin.), we resolve the quantities into their prime factors. The only prime factors common to both quantities are 3, aa, b, and x, and their product, $3\,a^2bx$, is the highest common divisor.

2. Find the H. C. D. of $a^4 - b^4$ and $a^6 + b^6$.

PROCESS

$$\begin{array}{l}a^4 - b^4 = (a^2 + b^2)(a + b)(a - b) \\ a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4) \\ \hline \text{H. C. D} = a^2 + b^2, \ \textit{Ans.}\end{array}$$

Therefore, to find the highest common divisor of two or more quantities we have the following

RULE. *Resolve the quantities into their prime factors, and take the product of all the common factors.*

EXAMPLES

Find the H. C. D. by inspection of:

3. $9\,x^2y^3z$ and $12\,xyz^2$.
4. $6\,m^2n^3$ and $18\,m^3n^2$.
5. $12\,m^2n^3y^5$ and $16\,m^3n^2y^3$.
6. $14\,l^2m^4n^6$ and $21\,l^6m^4n^2$.
7. $x^2 - y^2$ and $(x - y)^2$.
8. $x^2 - y^2$ and $(x + y)^2$.
9. $15\,xy^2z^3$ and $20\,x^2y^3z^2$.
10. $25\,(x - y)^2$ and $30\,(x^3 - y^3)$.

11. $30x^2y^2$ and $40x^3y$.

12. x^3+y^3 and $x^2+2xy+y^2$.

13. x^2+3x and x^2-9.

14. x^6+y^6 and $x^{10}+y^{10}$.

Find the H. C. D. by factoring of:

15. x^2-x-6 and $x^2-3x-10$.

16. $x^2+10x+21$ and $x^2-2x-15$.

17. x^2-81 and $x^2+4x-45$.

18. x^6-y^6 and x^9-y^9.

19. x^2-y^2, $(x-y)^2$, and x^2y-xy^2.

20. $(a+b)^2-c^2$ and $a^2-(b+c)^2$.

21. $a^2-(b-c)^2$ and $b^2-(a-c)^2$.

22. m^3+n^3, m^2-n^2, and $(m+n)^3$.

23. x^6+y^6, $x^{10}+y^{10}$, $x^4+2x^2y^2+y^4$.

24. m^2-16, $m^2+5m-36$, and m^2-2m-8.

25. b^2-c^2, $b^2+4bc+3c^2$, and $b^2-4bc-5c^2$.

26. x^2+x-12, x^2-x-20, and $x^2+8x+16$.

27. $mn+mc$, $n^2+2nc+c^2$, and n^3+c^3.

28. x^3+1, x^2-x+1, and $2x^3-2x^2+2x$.

29. $2m+6y$, $2(m^2+6my+9y^2)$, and $4am+12ay$.

30. m^2+m-20, m^2-m-12, and m^2-2m-8.

LOWEST COMMON MULTIPLE

47. A **Multiple** of a quantity is a quantity that will exactly contain it.

Thus, $3a^2b$ is a multiple of ab.

48. A **Common Multiple** of two or more quantities is a quantity that is a multiple of each of them.

Thus, $3a^2b$ is a common multiple of $3a$ and $3b$.

49. The **Lowest Common Multiple** (L. C. M.) of two or more quantities is a quantity of lowest degree that is a multiple of each of them.

Thus, $6ab$ is the L. C. M. of $3a$ and $2b$.

PRINCIPLE. *The lowest common multiple of two or more quantities must contain all the factors of each quantity, and no others.*

50. To find the lowest common multiple of quantities that can be readily factored.

1. Find the L. C. M. of $6a^2bc^2$ and $9ab^2c$.

PROCESS

$$\begin{array}{l} 6a^2bc^2 = 2 \times 3 \times aa \times b \times cc \\ 9ab^2c = 3 \times 3 \times a \times bb \times c \\ \hline \text{L. C. M.} = 9ab^2c \times 2 \times a \times c = 18a^2b^2c^2, \; \textit{Ans.} \end{array}$$

The L. C. M. must contain $9ab^2c$ (Art. 49, Def.) and all the factors of $6a^2bc^2$ not found in $9ab^2c$ (Art. 49, Prin.), which are 2, a, and c. Hence the L. C. M. $= 9ab^2c \times 2 \times a \times c = 18a^2b^2c^2$.

2. Find the L. C. M. of $a^4 - b^4$ and $a^6 + b^6$.

PROCESS

$$\begin{array}{l} a^4 - b^4 = (a^2 + b^2)(a + b)(a - b) \\ a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4) \\ \hline \text{L. C. M.} = (a^6 + b^6)(a + b)(a - b) = a^8 - a^6b^2 + a^2b^6 - b^8, \; \textit{Ans.} \end{array}$$

Therefore, to find the lowest common multiple of two or more quantities, we have the following

RULE. *Resolve the quantities into their prime factors. Multiply the quantity of highest degree by all the factors of the other quantities not found in the highest quantity.*

EXAMPLES

Find the L. C. M. by inspection of:

3. $4x^2y$, $12xy^2$, and $24x^3yz$.
4. $15ab^2c^3$, $30a^3bc^2$, and $60a^2b^3c$.
5. x^2+y^2, x^4-y^4, and x^8-y^8.
6. x^2-y^2, x^4+y^4, x^4-y^4, and x^8-y^8.
7. x^3-y^3, x^3+y^3, and x^6-y^6.
8. x^2-1, x^3+1, and x^3-1.
9. $3x+1$, $2(3x-1)$, and $4(9x^2-1)$.
10. x^2-1, x^3+1, and x^6-1.

Find the L. C. M. by factoring of:

11. y^2-5x+6 and y^2-6y+8.
12. x^2-9, x^2+x-12, and $x^2+2x-15$.
13. x^2+y^2, x^6+y^6, x^6-y^6, and $x^{12}-y^{12}$.
14. $x^2+6x-40$ and $x^2-10x+24$.
15. $(a+b)^2-c^2$, $(a+c)^2-b^2$, and $a^2-(b+c)^2$.
16. a^2+ab+b^2, a^2-ab+b^2, and a^6-b^6.
17. $(x+y)^2$, $(x-y)^2$, and x^2-y^2.
18. x^2+y^2, x^6+y^6, x^6-y^6, and $x^{12}-y^{12}$.
19. $2-2x+2x^2$, $3+3x+3x^2$, and $1+x^3$.
20. $x^2+2xy+y^2$, $x^2-2xy+y^2$, and $x^4-2x^2y^2+y^4$.
21. $n^2+3n-40$, n^2-64, and $n^2-13n+40$.
22. $a^2-19a+88$, $a^2-3a-88$, and a^2-64.
23. $m^2+7m+12$, m^2+6m+8, and m^2+5m+6.
24. x^2-1, x^3+x^2+x+1, and x^3-x^2+x-1.
25. $x^2+2xy+y^2$, $x^2-2xy+y^2$, and x^2-y^2.
26. a^2-4, a^2-5a+6, and $a^2+3a-10$.
27. $(m+n)^2-a^2$, $(m+n-a)$, and $(m+n+a)^2$.

IV. FRACTIONS

51. An **Algebraic Fraction** is an indicated expression of division.

Thus, $\frac{a}{b}$ is a fraction and is read a divided by b, and not *a over b*.

52. The dividend, a, is the **Numerator**, the divisor, b, is the **Denominator**; and the numerator and denominator are the **Terms** of the fraction.

53. An **Integral Quantity** is one that can be expressed without the fractional form; as $m+n$.

A **Mixed Quantity** is the sum or difference of an integer and a fraction; as $a+\frac{b}{c}$, or $a-\frac{b}{c}$.

54. PRINCIPLE I. *Multiplying the numerator or dividing the denominator multiplies the value of the fraction.*

PRINCIPLE II. *Dividing the numerator or multiplying the denominator divides the value of the fraction.*

PRINCIPLE III. *Multiplying or dividing both numerator and denominator by the same quantity does not change the value of the fraction.*

REDUCTION OF FRACTIONS

55. CASE I. **To reduce a fraction to its lowest terms.**

A fraction is in its *lowest terms* when the numerator and denominator have no common factor.

Since dividing both numerator and denominator by the same quantity does not change the value of the fraction (Art. 54, Prin. III), we have the following

RULE. *Resolve both terms of the fraction into their prime factors, and cancel all factors that are common to both terms.*

Reduce the following fractions to their lowest terms:

1. $\dfrac{12\,x^2y^3z^2}{16\,x^3y^4z} = \dfrac{\cancel{2}\times\cancel{2}\times 3\,\cancel{x^2}\cancel{y^3}z^2}{\cancel{2}\times\cancel{2}\times 2\times 2\times x^3y^4\cancel{z}} = \dfrac{3\,z}{4\,xy}.$

2. $\dfrac{x^2+7\,x+12}{x^2+9\,x+20} = \dfrac{\cancel{(x+4)}(x+3)}{\cancel{(x+4)}(x+5)} = \dfrac{x+3}{x+5}.$

EXAMPLES

Reduce to lowest terms:

3. $\dfrac{24\,m^4n^7}{36\,m^2n^8}.$

4. $\dfrac{36\,a^6b^4c^5}{72\,a^7b^8c^2}.$

5. $\dfrac{121\,m^3n^4w^2}{154\,m^2n^5w^2}.$

6. $\dfrac{33\,x^2y^4z^3}{44\,x^{10}y^2z^4}.$

7. $\dfrac{60\,m^4n^3p^4}{75\,m^9n^5p^2}.$

8. $\dfrac{60\,a^4b^3c}{72\,a^4b^2c}.$

9. $\dfrac{x^2y-xy^2}{2\,xy^2-3\,x^2y}.$

10. $\dfrac{4\,a^2+8\,a}{4\,a^2-8\,a}.$

11. $\dfrac{3\,x^2+6\,x}{6\,x^2-9\,x}.$

12. $\dfrac{x^2-1}{x^3+1}.$

13. $\dfrac{x^2-y^2}{(x-y)^2}.$

14. $\dfrac{a^2+17\,a+66}{a^2+18\,a+72}.$

15. $\dfrac{m^2-17\,m+72}{m^2-18\,m+80}.$

16. $\dfrac{a^3-b^3}{a^5-b^5}.$

17. $\dfrac{a^3+b^3}{(a+b)^3}.$

18. $\dfrac{(m+n)^2-a^2}{m^2-(n+a)^2}.$

19. $\dfrac{x^2-(y-z)^2}{(x+y)^2-z^2}.$

56. CASE II. **To reduce a fraction to an integral or a mixed quantity.**

Since a fraction is an indicated expression of division (Art. 51), we have the following

RULE. *Divide the numerator by the denominator.*

1. Reduce $\dfrac{a^3+b^3}{a+b}$ to an integral quantity.

By division $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2.$

2. Reduce $\dfrac{x^2+7x+14}{x+4}$ to a mixed number.

$$\begin{array}{l} x+4)\,x^2+7x+14\,(x+3+\dfrac{2}{x+4},\ \textit{Ans.} \\ \qquad\quad \underline{x^2+4x\qquad\quad} \\ \qquad\qquad\quad 3x+14 \\ \qquad\qquad\quad \underline{3x+12} \\ \qquad\qquad\qquad\quad 2 \end{array}$$

EXAMPLES

Reduce to integral or mixed quantities:

3. $\dfrac{x^3+y^3}{x-y}$.

4. $\dfrac{x^5+y^5}{x+y}$.

5. $\dfrac{m^6+n^6}{m^2+n^2}$.

6. $\dfrac{x^3-y^3}{x+y}$.

7. $\dfrac{a^3}{a-b}$.

8. $\dfrac{x^4+y^4}{x+y}$.

9. $\dfrac{x^2-a^2}{x-a}$.

10. $\dfrac{x^4+y^4}{x^2+y^2}$.

11. $\dfrac{a^3+b^3}{(a+b)^3}$.

12. $\dfrac{x^3+4x+6}{x+2}$.

13. $\dfrac{12x^3+8x^2+9}{4x^2+1}$.

14. $\dfrac{a^4-1}{a^2+a+1}$.

15. $\dfrac{5x^2+6x+7}{x+4}$.

16. $\dfrac{m^2-n^2-3}{m+n}$.

17. $\dfrac{x^4-x^3+2}{x^2-x-1}$.

57. CASE III. **To reduce a mixed quantity to a fraction.**

The process is precisely the same as in arithmetic; hence we have the following

RULE. *Multiply the integral part by the denominator of the fraction; to this product add the numerator when the sign before the fraction is +, and subtract it when the sign is −, and write the result over the denominator.*

1. Reduce $x+y-\dfrac{a^2-y^2}{x-y}$ to a fraction.

$$x+y-\frac{a^2-y^2}{x-y}=\frac{(x+y)(x-y)-(a^2-y^2)}{x-y}$$
$$=\frac{x^2-y^2-a^2+y^2}{x-y}$$
$$=\frac{x^2-a^2}{x-y}.$$

EXAMPLES

Reduce the following to fractional forms:

2. $1+\dfrac{x+y}{x-y}.$

3. $x-1-\dfrac{x-1}{x+1}.$

4. $a+b-\dfrac{(a-b)^2}{a+b}.$

5. $a+b-\dfrac{a^2+b^2}{a+b}.$

6. $x+y+\dfrac{x^2-y^2}{x+y}.$

7. $m+n-\dfrac{2\,mn}{m+n}.$

8. $4\,x+1-\dfrac{3\,x^2}{2\,x-1}.$

9. $x^2+xy+y^2-\dfrac{x^4+y^4}{x^2-xy+y^2}.$

10. $a^2+2\,ab+b^2-\dfrac{a^3-3\,a^2b+3\,ab^2+b^3}{a+b}.$

11. $x^2-x+1-\dfrac{x^3-1}{x+1}.$

58. Case IV. **To reduce fractions to their lowest common denominator.**

1. Reduce $\frac{x^2-1}{x^2+1}$ and $\frac{x^2+1}{x^2-1}$ to their lowest common denominator.

It is evident that the lowest common denominator is the lowest common multiple of the denominators x^2+1 and x^2-1, which is x^4-1 (Art. 50).

Since both terms of a fraction may be multiplied by the same quantity without changing its value (Art. 54, Prin. III), we multiply both terms of the first fraction by x^4-1 divided by x^2+1, which is x^2-1.

And $$\frac{x^2-1}{x^2+1}=\frac{(x^2-1)(x^2-1)}{(x^2+1)(x^2-1)}=\frac{(x^2-1)^2}{x^4-1}.$$

We multiply both terms of the second fraction by x^4-1 divided by x^2-1, which is x^2+1.

And $$\frac{x^2+1}{x^2-1}=\frac{(x^2+1)(x^2+1)}{(x^2-1)(x^2+1)}=\frac{(x^2+1)^2}{x^4-1}.$$

Therefore, the required fractions are

$$\frac{(x^2-1)^2}{x^4-1} \text{ and } \frac{(x^2+1)^2}{x^4-1}, \textit{ Ans.}$$

Hence, to reduce fractions to their lowest common denominator, we have the following

Rule. *Find the lowest common multiple of the denominators, divide this denominator by the denominator of each fraction, and multiply the numerator by the quotient.*

Note. Reduce each fraction to its lowest terms before beginning the process.

EXAMPLES

Reduce to equivalent fractions having the lowest common denominator:

2. $\frac{a}{b}$ and $\frac{b}{a}$.

3. $\frac{m}{n}$ and $\frac{x}{y}$.

4. $\frac{a}{4}$ and $\frac{2}{c}$.

5. $\frac{2b}{x^2y}$ and $\frac{3b}{xy^2}$.

6. $\frac{6a}{8ab}$ and $\frac{9b}{12ab}$.

7. $\frac{6m^2}{15m^3}$ and $\frac{8n^3}{20n^4}$.

8. $\frac{a}{mn}$, $\frac{b}{m^2}$, and $\frac{c}{mn^2}$.

9. $\frac{c}{4a}$, $\frac{6}{24c}$, and $\frac{4n}{6ac}$.

10. $\frac{4a}{2a^2c}$, $\frac{10c}{2bc^2}$, and $\frac{12ab}{3a^2b^2}$.

11. $\frac{a}{a+b}$, $\frac{b}{a-b}$, and $\frac{c}{a^2-b^2}$.

12. $\frac{m}{a^3-b^3}$, $\frac{n}{a-b}$, and $\frac{p}{a^2+ab+b^2}$.

13. $\frac{2}{2a+2}$, $\frac{3}{3a-3}$, and $\frac{4}{4a^2-4}$.

14. $\frac{2x}{x^2-xy+y^2}$, $\frac{3y}{x^3+y^3}$, and $\frac{4z}{x+y}$.

15. $\frac{1}{x^2+x-6}$, $\frac{2}{x^2+5x+6}$, and $\frac{3}{x^2-4}$.

16. $\frac{a}{x^2+xy+y^2}$, $\frac{b}{x^2-xy+y^2}$, and $\frac{c}{x^4+x^2y^2+y^4}$.

CLEARING EQUATIONS OF FRACTIONS

59. To clear an equation of fractions.

1. Solve the equation $\frac{2x}{5}-\frac{3}{10}=\frac{x}{15}+\frac{11}{30}$.

SOLUTION

Since both members of an equation may be multiplied by the same number without changing the equality (Art. 8, Axiom 3), we multiply both numbers of

$$\frac{2x}{5}-\frac{3}{10}=\frac{x}{15}+\frac{11}{30}$$

by the L.C.M. of the denominators, 5, 10, 15, and 30, which is 30, and we have

$$12x-9=2x+11.$$

Whence, $$10x=20,$$

$$x=2, \textit{ Ans.}$$

Therefore, to clear an equation of fractions we have the following

RULE. *Multiply each term of the equation by the lowest common multiple of the denominators.*

EXAMPLES

Solve the following equations:

2. $x+\frac{x}{5}=12.$
3. $2x+\frac{x}{3}=14.$
4. $\frac{x}{2}+\frac{x}{3}=10.$
5. $\frac{x}{4}+\frac{x}{5}=9.$
6. $\frac{x}{6}-\frac{x}{8}=1.$
7. $\frac{3x}{4}+\frac{x}{3}=13.$
8. $x+\frac{x}{2}+\frac{x}{3}=11.$
9. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13.$

10. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=47.$

11. $\frac{2x}{5}-\frac{3x}{7}+\frac{4x}{9}=131.$

12. $2x=\frac{x}{6}+\frac{x}{7}+71.$

13. $\frac{1}{x}+\frac{2}{x}=\frac{1}{2}.$

14. $\frac{2}{x}+\frac{3}{x}=\frac{4}{5}-\frac{7}{x}.$

15. $\frac{1}{12}-\frac{1}{x}+\frac{1}{3x}=\frac{1}{4}-\frac{1}{4x}.$

16. $\frac{x+3}{2}+\frac{x-3}{3}=\frac{4x+15}{6}.$

17. $\frac{3-x}{2}-\frac{4-x}{5}-\frac{x}{6}=4.$

18. $\frac{x-2}{2}+\frac{x-3}{3}=\frac{x-4}{4}+\frac{x-5}{5}+23.$

19. $\frac{x-2}{3}-\frac{x-3}{4}=\frac{x-4}{5}-\frac{x-5}{6}.$

20. $2x+\frac{x}{5}-\frac{3x}{7}=\frac{3x}{35}+59.$

21. $\frac{x+2}{4}-\frac{x+3}{5}=\frac{x+1}{2}-\frac{x+4}{3}.$

22. $\frac{x-1}{6}+\frac{x-3}{11}=\frac{x-4}{7}+\frac{x+2}{9}.$

23. $\begin{cases} x+y=5, \\ \frac{x}{3}+\frac{y}{2}=2. \end{cases}$

24. $\begin{cases} \frac{x}{3}+\frac{y}{4}=5, \\ x-y=1. \end{cases}$

25. $\begin{cases} \frac{x}{2}-\frac{y}{3}=14, \\ \frac{x}{3}-\frac{y}{4}=8. \end{cases}$

26. $\begin{cases} \frac{2}{3}x+\frac{3}{4}y=10, \\ \frac{1}{2}x-\frac{1}{4}y=1. \end{cases}$

27. $\begin{cases} \frac{x}{2}+\frac{3y}{2}=11, \\ \frac{x+y}{2}+\frac{x-y}{2}=2. \end{cases}$

28. $\begin{cases} \frac{x+y}{8}+\frac{x-y}{6}=10, \\ \frac{x+y}{4}-\frac{x-y}{3}=10. \end{cases}$

PROBLEMS

60. 1. What number increased by its $\frac{1}{2}$ gives a sum of 60?

2. What number diminished by its $\frac{1}{3}$ gives a difference of 40?

3. The sum of two numbers is 60, and $\frac{1}{4}$ of the greater minus $\frac{1}{5}$ of the less equals 6. What are the numbers?

4. A and B together have \$10,000, and $\frac{1}{4}$ of A's money plus $\frac{1}{2}$ B's equals \$4000. How much has each?

5. M and N had the same sum of money. M lost \$500 and N gained \$1000, whereupon it was found that $\frac{1}{5}$ of M's money equaled $\frac{1}{6}$ of N's. How much had each at first?

6. Find two consecutive numbers, x and $x+1$, such that $\frac{1}{7}$ of the smaller exceeds $\frac{1}{10}$ of the larger by 2.

7. The sum of two numbers is 84, and if the greater is divided by the smaller, the quotient is 3. Find the numbers.

NOTE. Let $x =$ the smaller number, and $84 - x$ the larger.

8. The sum of two numbers is 120, and if the larger is divided by the smaller, the quotient is 5. Find the numbers.

9. The difference of two numbers is 120, and if the larger is divided by the smaller, the quotient is 5. Find the numbers.

10. The difference of two numbers is 240, and if the larger is divided by the smaller, the quotient is 3. Find the numbers.

11. The sum of two numbers is 80, and if the larger is divided by the smaller, the quotient is 3 and the remainder is 4. Find the numbers.

NOTE. Let $x =$ the larger number, and $80 - x =$ the smaller. And since the

$$\frac{\text{Dividend} - \text{Remainder}}{\text{Divisor}} = \text{Quotient, then } \frac{x-4}{80-x} = 3.$$

12. The sum of two numbers is 150, and if the larger is divided by the smaller, the quotient is 4 and the remainder is 5. Find the numbers.

13. A teacher spent $\frac{1}{2}$ of his salary for board and clothes, and $\frac{1}{8}$ of the remainder for other expenses. What was his salary if he saved $300 a year?

14. How far can a person ride in a coach going at the rate of 10 miles an hour, provided he walks back at the rate of 4 miles an hour, and is gone 14 hours?

SUGGESTION. If $x =$ the distance he goes, how long will it take him to ride? To walk?

15. How far may a person ride on his bicycle, going at the rate of 15 miles an hour, provided he rides back in a car at the rate of 45 miles an hour, and is gone but 8 hours?

16. A boat whose rate of sailing in still water is 12 miles an hour descends a river whose current is 4 miles an hour. How far can it go and return, provided it is gone but 12 hours?

17. Arthur can mow a field in 10 days, Horace in 12 days, and Ross in 16 days. In what time can they together mow it?

18. Two pipes fill a cistern in 20 and 30 hours respectively. How long will it take to fill the cistern if both pipes are open at the same time?

ADDITION AND SUBTRACTION OF FRACTIONS

61. PRINCIPLE. *To add or subtract fractions, they must have a common denominator.*

It is evident $$\frac{a}{m}+\frac{b}{m}=\frac{a+b}{m}.$$

And $$\frac{a}{m}-\frac{b}{m}=\frac{a-b}{m}.$$

1. Add $\frac{4m}{an}$ and $\frac{4n}{am}$.

We must reduce the fractions to their lowest common denominator (Prin.). The L. C. D. is amn (Art. 59). Hence,

$$\frac{4m}{an}+\frac{4n}{am}=\frac{4m^2}{amn}+\frac{4n^2}{amn}$$

$$=\frac{4(m^2+n^2)}{amn}, \textit{ Ans.}$$

2. From $\frac{4x+7}{x-1}$ take $\frac{3x-2}{x+1}$.

The L. C. D. is x^2-1. Hence,

$$\frac{4x+7}{x-1}=\frac{(4x+7)(x+1)}{(x-1)(x+1)}=\frac{4x^2+11x+7}{x^2-1},$$

and $$\frac{3x-2}{x+1}=\frac{(3x-2)(x-1)}{(x+1)(x-1)}=\frac{3x^2-5x+2}{x^2-1};$$

then $$\frac{4x^2+11x+7}{x^2-1}-\frac{3x^2-5x+2}{x^2-1}=\frac{x^2+16x+5}{x^2-1}, \textit{ Ans.}$$

The preceding work may be arranged as follows:

The L. C. D. is x^2-1.

1st numerator is $(4x+7)(x+1)=4x^2+11x+7$

2d numerator is $(3x-2)(x-1)=\underline{3x^2-\ \ 5x+2}$

The difference of the numerators $=\ \ x^2+16x+5$

$\therefore$ the result is $$\frac{x^2+16x+5}{x^2-1}.$$

Therefore, to add or subtract fractions we have the following

RULE. *Reduce the fractions to their lowest common denominator, add or subtract their numerators, and write the result over the common denominator.*

NOTE. Always reduce the fractions to their lowest terms before reducing them to a common denominator.

EXAMPLES

Add the following:

3. $\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}.$

4. $\frac{m}{np}, \frac{p}{nm}, \frac{n}{pm}.$

5. $\frac{ac}{dc}, \frac{bc}{db}, \frac{bd}{d^2}.$

6. $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$

7. $\frac{1}{m+n}$ and $\frac{1}{m-n}.$

8. $\frac{a}{a+b}$ and $\frac{b}{a-b}.$

9. $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}.$

10. $\frac{m}{m^2-n^2}$ and $\frac{n}{m+n}.$

11. $\frac{1}{(x+1)(x+2)}$ and $\frac{1}{(x+2)(x+3)}.$

12. $\frac{m}{(m+2)(m+3)}$ and $\frac{n}{(m+2)(m+1)}.$

13. $\frac{x}{x-x^2}, \frac{y}{y+xy}$ and $\frac{2}{2-2x^2}.$

Subtract the following:

14. $\frac{b}{ac}$ from $\frac{a}{bc}.$

15. $\frac{1}{xy}$ from $\frac{1}{yz}.$

16. $\frac{n}{m-n}$ from $\frac{m}{m+n}.$

17. $\frac{(c-d)^2}{cd}$ from $\frac{(c+d)^2}{cd}.$

18. $\frac{a-b}{a+b}$ from $\frac{a+b}{a-b}.$

19. $\frac{1}{x+y}$ from $\frac{1}{x-y}.$

20. $\dfrac{1}{(x+1)(x+2)}$ from $\dfrac{2}{(x+1)(x+3)}$.

21. $\dfrac{1}{(x-1)(x-2)}$ from $\dfrac{2}{(x-1)(x-3)}$.

22. $\dfrac{3}{x^2+9x+18}$ from $\dfrac{4}{x^2+8x+12}$.

Find the value of the following:

23. $\dfrac{1}{x+3}+\dfrac{1}{x-2}$.

24. $\dfrac{1}{x+4}-\dfrac{1}{x+6}$.

25. $\dfrac{1}{x+y}+\dfrac{1}{x-y}$.

26. $\dfrac{1}{x-y}-\dfrac{1}{x+y}$.

27. $\dfrac{1}{x^2+x+1}+\dfrac{1}{x^2-x+1}$.

28. $\dfrac{2}{a^2+a+1}-\dfrac{2}{a^2-a+1}$.

29. $\dfrac{a^2}{a^3-b^3}-\dfrac{ab}{a^2+ab+b^2}+\dfrac{b}{a-b}$.

30. $\dfrac{x-5}{x^2-4}+\dfrac{8}{x+2}-\dfrac{3}{x-2}$.

31. $\dfrac{x-1}{x^2-5x+6}+\dfrac{x-2}{x^2-4x+3}-\dfrac{x-3}{x^2-3x+2}$.

32. $\dfrac{x^2-x+1}{x^3+1}+\dfrac{x^3+x^2+x+1}{x^4-1}-\dfrac{2(x^4+x^2+1)}{x^6-1}$.

Suggestion. Reduce to lowest terms.

33. $\dfrac{x-2}{x^2-x+1}+\dfrac{x^2-2x+3}{x^3+1}-\dfrac{1}{x+1}$.

MULTIPLICATION OF FRACTIONS

62. CASE I. **To multiply a fraction by an integer.**

1. Multiply $\frac{a}{b}$ by c and $\frac{a}{bc}$ by c.

Since multiplying the numerator of a fraction by any quantity multiplies the value of the fraction by that quantity (Art. 54, Prin. I), then

$$\frac{a}{b} \times c = \frac{ac}{b}.$$

Since dividing the denominator of a fraction by any quantity multiplies the value of the fraction by that quantity (Art. 54, Prin. I), then

$$\frac{a}{bc} \times c = \frac{a}{b}.$$

Therefore, to multiply a fraction by an integer, we have the following

RULE. *Multiply the numerator or divide the denominator of the fraction by the multiplier.*

It is frequently convenient to indicate the multiplication and cancel common factors from both terms.

2. Multiply $\frac{m+n}{m^2-2mn+n^2}$ by m^2-n^2.

$$\frac{m+n}{m^2-2mn+n^2} \times (m^2-n^2) = \frac{m+n}{(m-n)\cancel{(m-n)}} \times \cancel{(m-n)}(m+n)$$

$$= \frac{(m+n)(m+n)}{m-n} = \frac{(m+n)^2}{m-n}, \textit{ Ans.}$$

Multiply the following:

3. $\frac{m}{n}$ by x.

4. $\frac{ab}{c^2d}$ by ac.

5. $\frac{2ab}{3n^2m}$ by $9mn$.

6. $\frac{ab}{c}$ by b.

7. $\dfrac{am}{2n^2}$ by $4n$.

8. $\dfrac{2xy}{3a^2b}$ by $12ab^2$.

9. $\dfrac{4}{(a-y)^2}$ by a^2-y^2.

10. $\dfrac{4a}{x^4-y^4}$ by x^2-y^2.

11. $\dfrac{4x^2}{x^4-x}$ by x^2+x+1.

12. $\dfrac{a-y}{a^2-x^2}$ by $a^2+2ax+x^2$.

13. $\dfrac{2}{x^4+x^2+1}$ by x^2+x+1.

14. $\dfrac{3a}{(x-y)^3}$ by x^3-y^3.

63. Case II. **To multiply a fraction by a fraction.**

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$\frac{a}{b}\times c=\frac{ac}{b}$ (Case I); but we wish to multiply by $\frac{c}{d}$. Since the multiplier is divided by d, the product, $\frac{ac}{b}$, must be divided by d, which, by Art. 54, Prin. II, is $\frac{ac}{bd}$.

Hence, $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd},\ \textit{Ans.}$$

Therefore, to multiply a fraction by a fraction, we have the following

Rule. *Multiply the numerators together for the numerator, and the denominators for the denominator.*

2. Multiply $\dfrac{m^2-4}{m^2-mn}$, $\dfrac{m^2+m}{m^2-m-6}$, and $\dfrac{m^2-3m}{m^2-m-2}$.

PROCESS

$$\frac{\cancel{(m+2)}\cancel{(m-2)}}{\cancel{m}(m-n)}\times\frac{\cancel{m}\cancel{(m+1)}}{\cancel{(m-3)}\cancel{(m+2)}}\times\frac{m\cancel{(m-3)}}{\cancel{(m+1)}\cancel{(m-2)}}=\frac{m}{m-n},\ \textit{Ans.}$$

EXAMPLES

Multiply the following:

3. $\dfrac{3a^2b^2}{4m^2n}$ by $\dfrac{12mn}{9ab^2}$.

4. $\dfrac{6x^2y}{5ab^2c}$ by $\dfrac{10b^2c}{18xy}$.

5. $\dfrac{a+b}{c}$ by $\dfrac{ac+cx}{a^2-b^2}$.

6. $\dfrac{a+b}{(a-b)^2}$ by $\dfrac{a-b}{(a+b)^2}$.

7. $\dfrac{m^2-m}{nx}$ by $\dfrac{xn}{m^2-1}$.

8. $\dfrac{a^2-b^2}{(a-b)^2}$ by $\dfrac{a^2-ab}{a+b}$.

9. $\dfrac{a^3+b^3}{a^3-b^3}$ by $\dfrac{a-b}{a+b}$.

10. $\dfrac{m^3-n^3}{m^3+n^3}$ by $\dfrac{m^2-mn+n^2}{m^2+mn+n^2}$.

11. $\dfrac{x^2-5x+6}{x^2+x-20}$ by $\dfrac{x^2+2x-24}{x^2-9}$.

12. $\dfrac{a^2-b^2}{a^2+b^2}$, $\dfrac{a+b}{(a-b)^2}$, and $\dfrac{a^4-b^4}{(a+b)^2}$.

13. $\dfrac{a^2+2ab+b^2-c^2}{a^2-2ab+b^2-c^2}$ by $\dfrac{a-b+c}{a+b+c}$.

14. $\dfrac{a^4-y^4}{x^2+2xy+y^2}$ by $\dfrac{x+y}{a^3+a^2y+ay^2+y^3}$.

15. $\dfrac{x^2+7x+10}{x^2-7x+10}$, $\dfrac{x^2-9x+14}{x^2-3x-40}$, and $\dfrac{x^2-13x+40}{x^2-5x-14}$.

16. $\dfrac{x^4+x^2+1}{x^3-1}$, $\dfrac{x-1}{x^2+x+1}$, and $\dfrac{x^2+x+1}{x^2-x+1}$.

17. $\dfrac{a^6-b^6}{a^6+b^6}$, $\dfrac{a-b}{a^4-b^4}$, and $\dfrac{a^2+b^2}{a^3-b^3}$.

18. $\dfrac{m^6-n^6}{m^4+m^2n^2+n^4}$, $\dfrac{m^2+n^2}{m^2-mn+n^2}$, and $\dfrac{m+n}{m^3-n^3}$.

19. $\dfrac{x^2-4}{x^2-9}$, $\dfrac{x^2-16}{x^2-25}$, $\dfrac{x^2+2x-15}{x^2+2x-8}$, and $\dfrac{x^2-2x-15}{x^2-2x-8}$.

DIVISION OF FRACTIONS

64. CASE I. **To divide a fraction by an integer.**

1. Divide $\frac{ac}{b}$ by c, and $\frac{a}{b}$ by c.

Since dividing the numerator of a fraction by any quantity divides the value of the fraction by that quantity (Art. 54, Prin. II), then

$$\frac{ac}{b} \div c = \frac{a}{b}.$$

Since multiplying the denominator of a fraction by any quantity divides the value of the fraction by that quantity (Art. 54, Prin. II), then

$$\frac{a}{b} \div c = \frac{a}{bc}.$$

Therefore, to divide a fraction by an integer we have the following

RULE. *Divide the numerator or multiply the denominator of the fraction by the divisor.*

It is frequently convenient to indicate the division, and cancel common factors from both terms.

2. Divide $\frac{x^2-25}{x+4}$ by x^2+x-20.

$$\frac{x^2-25}{x+4} \div x^2+x-20 = \frac{\cancel{(x+5)}(x-5)}{(x+4)(x-4)\cancel{(x+5)}} = \frac{x-5}{x^2-16} \quad \textit{Ans.}$$

EXAMPLES

Divide the following:

3. $\frac{an}{b}$ by x.

4. $\frac{mn^2}{ax}$ by mn.

5. $\frac{ab^2c}{mn}$ by abc.

6. $\frac{12x^2y}{5ab}$ by $6xy$.

7. $\dfrac{7\,ax}{4\,mn}$ by $4\,mn$.

8. $\dfrac{18\,xy^2z}{5\,amn}$ by $mxyz$.

9. $\dfrac{x^3-y^3}{xy}$ by $x-y$.

10. $\dfrac{x^3+y^3}{x-y}$ by x^2-xy+y^2.

11. $\dfrac{a^3-1}{a+1}$ by a^2+a+1.

12. $\dfrac{x^5y-xy^5}{x^2y^2}$ by $x^3+x^2y+xy^2+y^3$.

13. $\dfrac{a^3+2\,a^2+a}{a-1}$ by $(a+1)^2$.

14. $\dfrac{x^4+x^2+1}{x+1.}$ by x^2-x+1.

15. $\dfrac{y^2+7\,y-30}{y+5}$ by $y^2+15\,y+50$.

65. CASE II. **To divide a fraction by a fraction.**

1. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

$\frac{a}{b}\div c=\frac{a}{bc}$ (Case I); but we wish to divide by $\frac{c}{d}$. Since the divisor is *divided* by d, the quotient, $\frac{a}{bc}$, must be *multiplied* by d, which, by Art. 54, Prin. I, gives $\frac{ad}{bc}$.

Hence, $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}, \text{ Ans.}$$

Therefore, to divide a fraction by a fraction, we have the following

RULE. *Invert the divisor, and proceed as in multiplication.*

2. Divide $\frac{x^2-25}{x^2-xy}$ by $\frac{x^2+3x-10}{xy-y^2}$.

PROCESS

$$\frac{x^2-25}{x^2-xy} \div \frac{x^2+3x-10}{xy-y^2} = \frac{\cancel{(x+5)}(x-5)}{x\cancel{(x-y)}} \times \frac{y\cancel{(x-y)}}{\cancel{(x+5)}(x-2)}$$

$$= \frac{y(x-5)}{x(x-2)}, \quad \textit{Ans.}$$

EXAMPLES

Divide the following:

3. $\frac{6x^2y^2}{8ab^2}$ by $\frac{3xy}{4ab}$.

4. $\frac{14mn^2}{9x^2y}$ by $\frac{7mn}{3xy}$.

5. $\frac{5am}{7xy}$ by $\frac{3cd}{4zx}$.

6. $\frac{(m-n)^2}{m+n}$ by $\frac{(m+n)^2}{m^2-n^2}$.

7. $\frac{a^3-b^3}{a^3+b^3}$ by $\frac{a+b}{a-b}$.

8. $\frac{x^2-1}{xy}$ by $\frac{x+1}{y}$.

9. $1-\frac{1}{x^2}$ by $1+\frac{1}{x}$.

10. $x^2-\frac{1}{x^2}$ by $x-\frac{1}{x}$.

11. $m^3-\frac{1}{m^3}$ by $m-\frac{1}{m}$.

12. $m^3+\frac{1}{m^3}$ by $m+\frac{1}{m}$.

13. $\frac{x^2+10x+21}{x^2+7x-18}$ by $\frac{x^2-2x-63}{x^2+12x+27}$.

14. $\frac{m^2+10m+16}{m^2-10m+21}$ by $\frac{m^2+7m+10}{m^2-9m+14}$.

15. $\frac{(x+2)^2-16}{(x+3)^2-25}$ by $\frac{(x+1)^2-25}{(x+2)^2-36}$.

16. $\dfrac{(a+b)^2-c^2}{a^2-(b+c)^2}$ by $\dfrac{(a-b)^2-c^2}{a^2-(b-c)^2}$.

17. $\left(\dfrac{m}{m-n}-\dfrac{n}{m+n}\right)$ by $\left(\dfrac{m^2}{m^2+n^2}+\dfrac{n^2}{m^2-n^2}\right)$.

18. $\left(x+\dfrac{xy}{x-y}\right)\left(x-\dfrac{xy}{x+y}\right)$ by $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)$.

COMPLEX FRACTIONS

66. A **Complex Fraction** is one whose numerator or denominator, or both, are fractional.

1. Reduce $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ to a simple fraction.

PROCESS

Multiply both numerator and denominator of the fraction by bd, the L. C. M. of the denominators (Art. 54, Prin. III), we have

$$\frac{\frac{a}{b}\times bd}{\frac{c}{d}\times bd}=\frac{ad}{bc}.$$

2. Reduce $\dfrac{\dfrac{m}{m+n}+\dfrac{n}{m-n}}{\dfrac{m}{m-n}-\dfrac{n}{m+n}}$ to a simple fraction.

The L. C. M. of $m+n$ and $m-n$ is m^2-n^2. Multiplying both terms of the fraction by m^2-n^2, we have

$$\frac{m(m-n)+n(m+n)}{m(m+n)-n(m-n)}=\frac{m^2+n^2}{m^2+n^2}=1, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to simple fractions:

3. $\dfrac{1-\dfrac{1}{m}}{1+\dfrac{1}{m}}$.

4. $\dfrac{a+\dfrac{b}{c}}{a-\dfrac{b}{c}}$.

5. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a+b}{ab}}$.

6. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{a+b}{ab}}$.

7. $\dfrac{\dfrac{b^2}{a}-n}{\dfrac{b^2}{a}+n}$.

8. $\dfrac{a+2+\dfrac{2}{m}}{a-2-\dfrac{2}{m}}$.

9. $\dfrac{\dfrac{1}{a+1}+1}{\dfrac{1}{a-1}-1}$.

10. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}$.

11. $\dfrac{\dfrac{1}{x+y}+\dfrac{1}{x-y}}{\dfrac{1}{x+y}-\dfrac{1}{x-y}}$.

12. $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$.

13. $\dfrac{1}{m-\dfrac{1}{m-\dfrac{1}{m-1}}}$.

14. $\dfrac{\dfrac{y}{x-y}+\dfrac{x}{x+y}}{\dfrac{y}{x+y}-\dfrac{x}{x-y}}$.

15. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}$.

16. $\dfrac{\dfrac{2}{3-x}+6-x}{\dfrac{24}{3-x}-7-x}$.

V. SIMULTANEOUS EQUATIONS

See Arts. 24, 25, 26, and 27.

ELIMINATION BY COMPARISON

67. Elimination by Comparison is the process of eliminating by finding an expression for the same unknown quantity from each equation and placing them equal to each other.

1. Solve the equations $\begin{cases} 3x + 7y = 134, & (1) \\ 4x - 2y = 20. & (2) \end{cases}$

From (1), $$x = \frac{134 - 7y}{3}. \qquad (3)$$

From (2), $$x = \frac{20 + 2y}{4}. \qquad (4)$$

From (3) and (4),

$$\frac{134 - 7y}{3} = \frac{20 + 2y}{4}. \quad \text{(Ax. 7)} \qquad (5)$$

Clearing of fractions,

$$536 - 28y = 60 + 6y, \qquad (6)$$

or $$-34y = -476, \qquad (7)$$

$$y = 14.$$

Substituting the value of y in (3),

$$x = \frac{134 - 98}{3} = 12.$$

$$x = 12,\ y = 14,\ \textit{Ans.}$$

Therefore, to eliminate by comparison, we have the following

RULE. *Find an expression for the same unknown quantity from each equation, place these values equal to each other, and solve the resulting equation.*

EXAMPLES

Solve the following equations by comparison:

2. $\begin{cases} 4x - y = 10, \\ x + y = 5. \end{cases}$

3. $\begin{cases} 2x + 3y = 23, \\ 3x + 2y = 22. \end{cases}$

4. $\begin{cases} 3x - y = 14, \\ 8x + 3y = 60. \end{cases}$

5. $\begin{cases} x - y = 2, \\ 3x + 7y = 46. \end{cases}$

6. $\begin{cases} 2x + 3y = 14, \\ 4x - y = 14. \end{cases}$

7. $\begin{cases} 2x - 4y = 6, \\ 6x + 3y = 48. \end{cases}$

8. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 6, \\ \frac{1}{3}x - \frac{1}{9}y = 1. \end{cases}$

9. $\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = 10, \\ \dfrac{5x}{6} + \dfrac{7y}{8} = 12. \end{cases}$

10. $\begin{cases} .2x + .3y = 8, \\ .3x + .2y = 7. \end{cases}$

11. $\begin{cases} x + y = 20, \\ 4x - 5y = 8. \end{cases}$

12. $\begin{cases} 7x + 3y = 58, \\ 3x - 5y = 6. \end{cases}$

13. $\begin{cases} x + y = 17, \\ x - y = 7. \end{cases}$

14. $\begin{cases} 3x - 2y = 11, \\ 5x + 2y = 29. \end{cases}$

15. $\begin{cases} 6x - 7y = -9, \\ 8x + 3y = 25. \end{cases}$

16. $\begin{cases} 5x + 3y = 55, \\ 11x + 10y = 155. \end{cases}$

17. $\begin{cases} x - y = 1, \\ 7x + 7y = 35. \end{cases}$

18. $\begin{cases} \frac{1}{4}x + \frac{1}{5}y = 10, \\ \frac{1}{6}x + \frac{1}{7}y = 7. \end{cases}$

19. $\begin{cases} \frac{2}{5}x - \frac{2}{5}y = 6, \\ \frac{1}{3}x - \frac{1}{2}y = 10. \end{cases}$

ELIMINATION BY SUBSTITUTION

68. Elimination by Substitution is the process of eliminating by finding an expression for the value of an unknown quantity from one equation, and substituting it in the other.

1. Solve the equations $\begin{cases} 2x + 3y = 32, & (1) \\ 3x + 2y = 28. & (2) \end{cases}$

From (1), $$x = \frac{32 - 3y}{2}. \qquad (3)$$

Substituting this value in (2),

$$3\left(\frac{32 - 3y}{2}\right) + 2y = 28. \qquad (4)$$

Clearing of fractions, $96 - 9y + 4y = 56.$

Or, $$-5y = -40,$$

$$y = 8.$$

Substituting this value of y in (3),

$$x = \frac{32 - 24}{2} = 4. \qquad x = 4,\ y = 8,\ \textit{Ans.}$$

Therefore, to eliminate by substitution, we have the following

RULE. *Find the value of one of the unknown quantities from either equation, substitute this value for the same unknown quantity in the other equation, and solve the resulting equation.*

EXAMPLES

Solve the following equations by substitution:

2. $\begin{cases} x + 2y = 5, \\ 2x + y = 4. \end{cases}$

3. $\begin{cases} 7x - 5y = 9, \\ 3x + 10y = 16. \end{cases}$

4. $\begin{cases} x - y = 1, \\ 5x + 7y = 29. \end{cases}$

5. $\begin{cases} 6x - 3y = 2, \\ 2x - 3y = 0. \end{cases}$

6. $\begin{cases} x+\dfrac{y}{2}=5, \\ 2x+3y=18. \end{cases}$

7. $\begin{cases} 3x+y=25, \\ x+3y=35. \end{cases}$

8. $\begin{cases} x+4y=25, \\ y+4x=10. \end{cases}$

9. $\begin{cases} 6x-y=22, \\ x+11y=26. \end{cases}$

10. $\begin{cases} 8x-7y=0, \\ 4x+3y=52. \end{cases}$

11. $\begin{cases} 2x+\ \ y=24, \\ 6x+7y=120. \end{cases}$

12. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=13, \\ \frac{1}{4}x+\frac{1}{5}y=7. \end{cases}$

13. $\begin{cases} x+y=30. \\ 2x-y=30. \end{cases}$

14. $\begin{cases} 4x-5y=0, \\ 7x+2y=43. \end{cases}$

15. $\begin{cases} 10x-y=9, \\ 10y-x=108. \end{cases}$

16. $\begin{cases} 5x=11y-117, \\ 7x=93-6y. \end{cases}$

17. $\begin{cases} .1x+.01y=2, \\ .3x-.01y=2. \end{cases}$

LITERAL SIMULTANEOUS EQUATIONS

69. It will generally be found most convenient in solving literal simultaneous equations to eliminate by addition and subtraction (Art. 27).

1. Solve the equations $\begin{cases} ax+by=c. & (1) \\ mx+ny=d. & (2) \end{cases}$

Multiplying (1) by m, $\quad amx+bmy=cm.$ (3)

Multiplying (2) by a, $\quad amx+any=ad.$ (4)

Subtracting (3) from (4), $\quad any-bmy=ad-cm.$ (5)

$$\therefore y=\frac{ad-cm}{an-bm}, \textit{ Ans.}$$

Multiplying (1) by n, $\quad anx+bny=cn.$ (6)

Multiplying (2) by b, $\quad bmx+bny=bd.$ (7)

Subtracting (7) from (6), $\quad anx-bmx=cn-bd.$

$$\therefore x=\frac{cn-bd}{an-bm}, \textit{ Ans.}$$

EXAMPLES

Solve the following:

2. $\begin{cases} x+y=a, \\ x-y=b. \end{cases}$

3. $\begin{cases} x+y=m+n, \\ x-y=m-n. \end{cases}$

4. $\begin{cases} ax+by=m, \\ bx+ay=n. \end{cases}$

5. $\begin{cases} mx+ny=a, \\ x-y=b. \end{cases}$

6. $\begin{cases} \dfrac{x}{m}-\dfrac{y}{n}=1, \\ \dfrac{x}{n}+\dfrac{y}{n}=1. \end{cases}$

7. $\begin{cases} \dfrac{ax+by}{2}=m, \\ \dfrac{ax-by}{2}=n. \end{cases}$

MISCELLANEOUS EXAMPLES

70. Reduce the following equations to their simplest form, and apply the method best adapted to each example.

Solve the following:

1. $\begin{cases} \dfrac{5x-6y}{2}+\dfrac{5y}{3}=13, \\ \dfrac{5x}{6}-\dfrac{7x-4y}{3}=7. \end{cases}$

2. $\begin{cases} \dfrac{x+4}{3}+4y=16, \\ \dfrac{x+2y}{2}+3x=31. \end{cases}$

3. $\begin{cases} \dfrac{x+2}{3}+\dfrac{y+3}{2}=7, \\ \dfrac{x+y}{6}+x=9. \end{cases}$

4. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=m, \\ \dfrac{x}{b}+\dfrac{y}{a}=n. \end{cases}$

5. $\begin{cases} \dfrac{x+2}{y+3}=\dfrac{x-2}{y-3}, \\ \dfrac{3x-5}{2y-2}=\dfrac{6x-14}{4y-2}. \end{cases}$

6. $\begin{cases} \dfrac{x+y}{x-y}=\frac{7}{8}, \\ \dfrac{2x-6}{3x-5}=\frac{4}{5}. \end{cases}$

7. $\begin{cases} \dfrac{x+y}{4}+\dfrac{x-y}{3}=6\frac{1}{3}, \\ 2x+3y=48. \end{cases}$

8. $\begin{cases} \dfrac{x+y}{x-y}=\frac{3}{4}, \\ \dfrac{x+y+2}{x-y-2}=\frac{3}{7}. \end{cases}$

9. $\left\{ \dfrac{x+2}{4} = \dfrac{y+3}{6} = \dfrac{x+y}{5}. \right.$

10. $\left\{ \begin{array}{l} \dfrac{4}{3x+4y-2} = \dfrac{3}{3y+4x-5}, \\ \dfrac{5}{6x+7y+2} = \dfrac{3}{13x-4y+7}. \end{array} \right.$

11. $\left\{ \begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = a, \\ \dfrac{1}{x} - \dfrac{1}{y} = b. \end{array} \right.$

12. $\left\{ \begin{array}{l} \dfrac{2}{x} + \dfrac{5}{y} = 14, \\ \dfrac{3}{x} - \dfrac{2}{y} = 2. \end{array} \right.$

SUGGESTION. When the unknown quantities occur in the denominators, it is better, generally, not to clear of fractions.

13. $\left\{ \begin{array}{l} \dfrac{5}{x} + \dfrac{6}{y} = 33, \\ \dfrac{8}{x} - \dfrac{5}{y} = 9. \end{array} \right.$

14. $\left\{ \begin{array}{l} \dfrac{7}{x} - \dfrac{3}{y} = 19, \\ \dfrac{3}{x} + \dfrac{4}{y} = 24. \end{array} \right.$

15. $\left\{ \begin{array}{l} \dfrac{m}{x} + \dfrac{n}{y} = a, \\ \dfrac{n}{x} - \dfrac{m}{y} = b. \end{array} \right.$

16. $\left\{ \begin{array}{l} \dfrac{a}{x} + \dfrac{b}{y} = 2, \\ \dfrac{b}{x} + \dfrac{a}{y} = 3. \end{array} \right.$

SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES

71. Where there are three or more unknown quantities and a like number of equations, the solution may be effected as follows:

1. Solve the equations $\left\{ \begin{array}{ll} x + 2y + 3z = 14, & (1) \\ 2x + 3y + 4z = 20, & (2) \\ 3x + 4y + 6z = 29. & (3) \end{array} \right.$

$$
\begin{array}{lrr}
\text{Mult. (1) by 2,} & 2x+4y+6z=28 & (4) \\
\text{Subt. (2) from (4),} & \underline{2x+3y+4z=20} & (5) \\
 & y+2z=\ \ 8 & (6) \\
\text{Mult. (1) by 3,} & 3x+6y+9z=42 & (7) \\
\text{Subt. (3) from (7),} & \underline{3x+4y+6z=29} & (8) \\
 & 2y+3z=13 & (9) \\
\text{Mult. (6) by 2,} & \underline{2y+4z=16} & (10) \\
\text{Subt. (9) from (10),} & z=\ \ 3 & \\
\text{Substitute in (6),} & y+6=\ \ 8. & \\
\text{Whence,} & y=\ \ 2. & \\
\text{Substitute in (1),} & x+4+9=14. & \\
\text{Whence,} & x=\ \ 1. &
\end{array}
$$

$x=1,\ y=2,\ z=3,$ *Ans.*

Solve the following equations:

2. $\begin{cases} x+y-z=2, \\ x-y+z=6, \\ y-x+z=10. \end{cases}$

3. $\begin{cases} 3x+2y+z=13, \\ 2x+y-2z=5, \\ x+2y+5z=13. \end{cases}$

4. $\begin{cases} 2x-3y+4z=26, \\ 2x+3y-4z=22, \\ 2x+3y+4z=86. \end{cases}$

5. $\begin{cases} 2x+2y-z=9, \\ 3y+3z-x=33, \\ 4x+4z-y=35. \end{cases}$

6. $\begin{cases} 3x+4y-z=20, \\ 3y+4z-x=32, \\ 3z+4x-y=38. \end{cases}$

7. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}=23, \\ \dfrac{x}{3}+\dfrac{y}{4}-\dfrac{z}{6}=4, \\ \dfrac{x}{4}-\dfrac{y}{3}+\dfrac{z}{2}=13. \end{cases}$

8. $\begin{cases} 2x+3z=21, \\ 4y-z=3, \\ 3x+2z=19. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=3, \\ \dfrac{1}{y}+\dfrac{1}{z}=2, \\ \dfrac{1}{x}+\dfrac{1}{z}=4. \end{cases}$

10. $\begin{cases} 8x-3y=22, \\ 9y+z=61, \\ 7z-2x=39. \end{cases}$

11. $\begin{cases} x-y=1, \\ y+z=1, \\ x-z=1. \end{cases}$

12. $\begin{cases} x+y+z=a, \\ x-y-z=b, \\ x+y-z=c. \end{cases}$

13. $\begin{cases} x+y-z=0, \\ y+z-x=40, \\ z+x-y=20. \end{cases}$

14. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{a}+\dfrac{z}{a}=1, \\ \dfrac{x}{b}+\dfrac{y}{b}-\dfrac{z}{b}=1, \\ \dfrac{x}{c}-\dfrac{y}{c}+\dfrac{z}{c}=1. \end{cases}$

15. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=3, \\ \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=2, \\ \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=1. \end{cases}$

See Art. 70, suggestion to Problem 11.

16. $\dfrac{x+y}{5}=\dfrac{x+z}{6}=\dfrac{z+y}{7}=2.$

17. $\dfrac{x+y}{a}=\dfrac{x+z}{b}=\dfrac{z+y}{c}=2.$

18. $\dfrac{x-y}{a}=\dfrac{y-z}{b}=\dfrac{x+z}{c}=a.$

PROBLEMS PRODUCING EQUATIONS CONTAINING ONE OR MORE UNKNOWN QUANTITIES

72. 1. How far may a person ride in a car going at the rate of 30 miles an hour if he returns at the rate of 18 miles an hour and is gone but 10 hours?

2. A general formed his men into a solid square, and had 120 men over: he increased the side of the square by 4 men, and then lacked 240 men to complete the square. How many men were there in the army?

3. The fore wheels of a wagon are 10 feet, and the hind wheels 12 feet, in circumference. How far must the wagon go in order that the fore wheels may make 264 revolutions more than the hind wheels?

4. A man buys two kinds of tea, one at 60 cents a pound and the other at 90 cents a pound. How much of each kind must he take to make a mixture of 120 pounds worth 80 cents a pound?

5. If 6 pounds of sugar and 10 pounds of tea cost $6.30, and at the same price 10 pounds of sugar and 6 pounds of tea cost $4.10, what is the price of each per pound?

6. A drover sold 8 horses and 14 cows for $1060, and at the same rate 12 horses and 20 cows for $1560. What was the price of each per head?

7. A boy bought 12 oranges and 10 bananas for 56 cents; at another time at the same price 8 oranges and 15 bananas for 54 cents. What was the cost of each?

8. A fruit dealer sold 8 lemons, 6 oranges, and 5 bananas for 58 cents; 9 lemons, 8 oranges, and 10 bananas for 79 cents; and 10 of each kind for 90 cents. What was the price of each?

9. A and B are 280 rods apart: if they approach each other, they will meet in 20 minutes; but if they travel in the same direction, A will overtake B in 2 hours and 20 minutes. What is the speed of each per minute?

10. A and B can do a piece of work in 12 days, B and C in 16 days, and A and C in 20 days. How long will it take each to do it?

VI. INVOLUTION AND EVOLUTION

INVOLUTION

73. Involution is the process of raising a quantity to any required power.

Review the definition

1. **Of Power of a Quantity** (Art. 4).
2. **Of Exponent of a Quantity** (Art. 4).

74. PRINCIPLE I. *All powers of a positive quantity are positive.*

This is evident since the product of like signs gives plus.

PRINCIPLE II. *The even powers of a negative quantity are positive, and the odd powers are negative.*

This is evident, since

$$(-a)^2 = (-a)(-a) \qquad = +a^2,$$

$$(-a)^3 = (-a)(-a)(-a) \qquad = -a^3,$$

$$(-a)^4 = (-a)(-a)(-a)(-a) \qquad = +a^4,$$

$$(-a)^5 = (-a)(-a)(-a)(-a)(-a) = -a^5, \text{ etc.}$$

INVOLUTION OF MONOMIALS

75. 1. Find the value of $(2\,a^2x^3)^3$.

$$(2\,a^2x^3)^3 = 2\,a^2x^3 \times 2\,a^2x^3 \times 2\,a^2x^3 = 8\,a^6x^9, \textit{ Ans.}$$

SECOND METHOD

77. Arranging the terms of the result in Ex. 1, Art. 76, in another order, and factoring, we have

$$(a+b+c)^2 = a^2 + (2a+b)b + [2(a+b)+c]c.$$

1. Square $x+y+z$.

$$(x+y+z)^2 = x^2 + (2x+y)y + [2(x+y)+z]z.$$

Let the pupil state the rule.

Square:

2. $m+n$
3. $a+b$.
4. $t+u$.
5. $h+t+u$.
6. $m+n-p+q$.
7. $x-y+z-u$.

SPECIAL METHODS OF CUBING POLYNOMIALS

FIRST METHOD

78. 1. Cube $a+b$.

By squaring (Art. 76),

$$\begin{array}{rl} (a+b)^2 = & a^2 + 2ab + b^2 \\ & a + b \\ \hline & a^3 + 2a^2b + ab^2 \\ & \phantom{a^3 + {}} a^2b + 2ab^2 + b^3. \\ \hline & a^3 + 3a^2b + 3ab^2 + b^3. \end{array}$$

Therefore, to cube a polynomial we have the following

RULE. *The cube of a polynomial having positive terms equals the cube of the first term, plus three times the square of the first by the second, plus three times the first by the square of the second, plus the cube of the second, plus three times the*

SPECIAL METHODS OF SQUARING POLYNOMIALS

FIRST METHOD

76. Polynomials may be squared by multiplication; but the result may be written without performing the actual multiplication.

1. Square $a+b+c$.

By multiplication,

$$\begin{array}{llllll}
a + & b + & c & & & \\
a + & b + & c & & & \\
\hline
a^2 + & ab + & ac & & & \\
+ & ab & & + b^2 + & bc & \\
 & + & ac & + & bc + c^2 & \\
\hline
\multicolumn{6}{l}{a^2 + 2\,ab + 2\,ac + b^2 + 2\,bc + c^2}
\end{array}$$

This result may be arranged thus:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

From this arrangement we derive the following

RULE. *The square of a polynomial equals the square of each term, together with twice the product of the terms taken two and two.*

EXAMPLES

Square:

2. $a+b+c+d.$

3. $x+y+z+u.$

4. $h+t+u.$

5. $x-y+z-u.$

6. $m+n-p+q-o.$

7. $l-m+p-q+s.$

8. $a+l+g+e+b+r.$

9. $x-n-m-p+z.$

SECOND METHOD

77. Arranging the terms of the result in Ex. 1, Art. 76, in another order, and factoring, we have

$$(a+b+c)^2 = a^2 + (2a+b)b + [2(a+b)+c]c.$$

1. Square $x+y+z$.

$$(x+y+z)^2 = x^2 + (2x+y)y + [2(x+y)+z]z.$$

Let the pupil state the rule.

Square:

2. $m+n$
3. $a+b$.
4. $t+u$.
5. $h+t+u$.
6. $m+n-p+q$.
7. $x-y+z-u$.

SPECIAL METHODS OF CUBING POLYNOMIALS

FIRST METHOD

78. 1. Cube $a+b$.

By squaring (Art. 76),

$$\begin{array}{rl} (a+b)^2 = & a^2 + 2ab + b^2 \\ & a + \quad b \\ \hline & a^3 + 2a^2b + ab^2 \\ & \qquad\quad a^2b + 2ab^2 + b^3. \\ \hline & a^3 + 3a^2b + 3ab^2 + b^3. \end{array}$$

Therefore, to cube a polynomial we have the following

Rule. *The cube of a polynomial having positive terms equals the cube of the first term, plus three times the square of the first by the second, plus three times the first by the square of the second, plus the cube of the second, plus three times the*

square of the sum of the first and second by the third, plus three times the sum of the first and second by the square of the third, plus the cube of the third; and so on.

EXAMPLES

Cube:

2. $x+y$.
3. $m+x$.
4. $2m+3n$.
5. $a+b+c$.
6. $a+b+c+d$.
7. $h+t+u$.
8. $x+y+z+u$.
9. $l+m+n+o+p$.

SECOND METHOD

79. A second method is derived from the first method by factoring.

Thus, $(a+b+c)^3 = a^3 + (3a^2+3ab+b^2)b$

$$+ [3(a+b)^2 + 3(a+b)c + c^2]c.$$

Let the pupil derive the rule.

Cube:

1. $t+u$.
2. $h+t+u$.
3. $T+h+t+u$.
4. $a+b+c+d$.
5. $m+n+o+p$.
6. $l+a+n+d+s$.

THE BINOMIAL THEOREM

80. The **Binomial Theorem** is a general method for raising a binomial to any required power. It is a ready method of obtaining the powers of a binomial without the tedious process of multiplication.

Mult. (1) by 2,	$2x+4y+6z=28$	(4)
Subt. (2) from (4),	$\underline{2x+3y+4z=20}$	(5)
	$y+2z=\ \ 8$	(6)
Mult. (1) by 3,	$3x+6y+9z=42$	(7)
Subt. (3) from (7),	$\underline{3x+4y+6z=29}$	(8)
	$2y+3z=13$	(9)
Mult. (6) by 2,	$\underline{2y+4z=16}$	(10)
Subt. (9) from (10),	$z=\ \ 3$	
Substitute in (6),	$y+6=\ \ 8.$	
Whence,	$y=\ \ 2.$	
Substitute in (1),	$x+4+9=14.$	
Whence,	$x=\ \ 1.$	

$x=1,\ y=2,\ z=3,$ *Ans.*

Solve the following equations:

2. $\begin{cases} x+y-z=2, \\ x-y+z=6, \\ y-x+z=10. \end{cases}$

3. $\begin{cases} 3x+2y+z=13, \\ 2x+y-2z=5, \\ x+2y+5z=13. \end{cases}$

4. $\begin{cases} 2x-3y+4z=26, \\ 2x+3y-4z=22, \\ 2x+3y+4z=86. \end{cases}$

5. $\begin{cases} 2x+2y-z=9, \\ 3y+3z-x=33, \\ 4x+4z-y=35. \end{cases}$

6. $\begin{cases} 3x+4y-z=20, \\ 3y+4z-x=32, \\ 3z+4x-y=38. \end{cases}$

7. $\begin{cases} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=23, \\ \frac{x}{3}+\frac{y}{4}-\frac{z}{6}=4, \\ \frac{x}{4}-\frac{y}{3}+\frac{z}{2}=13. \end{cases}$

8. $\begin{cases} 2x+3z=21, \\ 4y-z=3, \\ 3x+2z=19. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=3, \\ \dfrac{1}{y}+\dfrac{1}{z}=2, \\ \dfrac{1}{x}+\dfrac{1}{z}=4. \end{cases}$

10. $\begin{cases} 8x-3y=22, \\ 9y+z=61, \\ 7z-2x=39. \end{cases}$

11. $\begin{cases} x-y=1, \\ y+z=1, \\ x-z=1. \end{cases}$

12. $\begin{cases} x+y+z=a, \\ x-y-z=b, \\ x+y-z=c. \end{cases}$

13. $\begin{cases} x+y-z=0, \\ y+z-x=40, \\ z+x-y=20. \end{cases}$

14. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{a}+\dfrac{z}{a}=1, \\ \dfrac{x}{b}+\dfrac{y}{b}-\dfrac{z}{b}=1, \\ \dfrac{x}{c}-\dfrac{y}{c}+\dfrac{z}{c}=1. \end{cases}$

15. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=3, \\ \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=2, \\ \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=1. \end{cases}$

See Art. 70, suggestion to Problem 11.

16. $\dfrac{x+y}{5}=\dfrac{x+z}{6}=\dfrac{z+y}{7}=2.$

17. $\dfrac{x+y}{a}=\dfrac{x+z}{b}=\dfrac{z+y}{c}=2.$

18. $\dfrac{x-y}{a}=\dfrac{y-z}{b}=\dfrac{x+z}{c}=a.$

PROBLEMS PRODUCING EQUATIONS CONTAINING ONE OR MORE UNKNOWN QUANTITIES

72. 1. How far may a person ride in a car going at the rate of 30 miles an hour if he returns at the rate of 18 miles an hour and is gone but 10 hours?

Therefore, to find any root of a monomial,

RULE. *Extract the required root of the coefficient, divide the exponent of each letter by the index of the root, and prefix to the result the proper sign.*

EXAMPLES

Find the values of the following:

2. $\sqrt{25\,a^2b^4}$.

3. $\sqrt[3]{-64\,a^6b^9}$.

4. $\sqrt{49\,m^2n^4x^6}$.

5. $\sqrt[4]{256\,a^4b^8c^{12}}$.

6. $\sqrt[5]{-32\,m^5n^{10}x^{15}}$.

7. $\sqrt[n]{a^{2n}b^{3n}c^{5n}x^{4n}}$.

8. $\sqrt[3]{-512\,a^{12}n^9m^6x^3}$.

9. $\sqrt[6]{a^{6n}b^{12m}c^{36}d^{24n}}$.

10. $\sqrt{\dfrac{16\,x^2y^4}{36\,a^4b^6}}$.

11. $\sqrt[3]{\dfrac{8\,m^3n^6}{27\,a^9b^{12}}}$.

12. $\sqrt[5]{-\dfrac{a^5b^{10}}{32\,x^{10}y^{15}}}$.

13. $\sqrt[4]{\dfrac{a^8b^{12}}{625\,m^4}}$.

14. Show that $\sqrt{\sqrt[3]{64}} = \sqrt[3]{\sqrt{64}}$.

15. Show that $\sqrt[3]{\sqrt[4]{a^{24}b^{48}}} = \sqrt[4]{\sqrt[3]{a^{24}b^{48}}}$.

SQUARE ROOT OF POLYNOMIALS

84. 1. Find the square root of $a^2 + 2\,ab + b^2$.

$$\begin{array}{l|l|l}
 & a^2 + 2\,ab + b^2 & a + b, \\
 & a^2 & \quad \textit{Ans.} \\
\hline
2\,a + b & 2\,ab + b^2 & \\
 & 2\,ab + b^2 & \\
\hline
\end{array}$$

From the method of squaring a polynomial (Art. 76), the first term, a^2, is the square of the first term of the root; hence the first term of the root is the square root of a^2, which is a. Subtracting its square from the polynomial, we have $2\,ab + b^2$ remaining.

The first term of this remainder is equal to twice the first term of the root by the second; hence we can find the second term of the

root by dividing $2ab$ by $2a$, which is b. Adding b to $2a$ gives $2a+b$, the complete divisor. Multiplying $2a+b$ by b and subtracting, there is no remainder. Hence, the square root of $a^2+2ab+b^2$ is $a+b$.

Therefore to extract the square root of a polynomial we have the following

RULE. *Arrange the terms according to the powers of some letter.*

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the polynomial.

Divide the first term of the remainder by twice the root already found, and annex the quotient to the root, and also to the trial divisor to form the complete divisor. Multiply the complete divisor by the second term of the root, and subtract the product from the remainder.

Continue in this manner until all the terms of the root are found.

2. Find the square root of:

$$
\begin{array}{l|l|l}
 & 4x^4-12x^3+13x^2-6x+1 & \underline{2x^2-3x+1}, \ \textit{Ans.} \\
 & 4x^4 & \\
\hline
4x^2-3x & -12x^3+13x^2 & \\
 & -12x^3+\ \ 9x^2 & \\
\hline
4x^2-6x+1 & \qquad 4x^2-6x+1 & \\
 & \qquad \underline{4x^2-6x+1} & \\
\end{array}
$$

EXAMPLES

Find the square root of the following:

3. x^2+2x+1.

4. $m^2+2mn+n^2$.

5. y^2+4y+4.

6. $9x^2+12x+4$.

7. $x^2-2xy+y^2$.

8. $16x^2-24xy+9y^2$.

9. $a^{2n}-2a^nb^n+b^{2n}$.

10. $x^{4n}+2x^{2n}y^{2n}+y^{4n}$.

11. $m^4 + 4m^3n + 10m^2n^2 + 12mn^3 + 9n^4$.

12. $x^4 - 2x^3 - 3x^2 + 4x + 4$.

13. $m^6 - 2m^5 - m^4 + 3m^2 + 2m + 1$.

14. $10n^2 + 25n^4 - 20n^3 + 1 - 4n + 16n^6 - 24n^5$.

15. $a^6 - 4a^5 + 14a^4 - 14a^3 + 13a^2 + 30a + 9$.

SQUARE ROOT OF NUMBERS

85. PRINCIPLE I. *The square of a number consists of twice as many figures as the number, or twice as many less one.*

$$1^2 = 1, \qquad 10^2 = 100, \qquad 100^2 = 10000,$$
$$9^2 = 81, \qquad 99^2 = 9801, \qquad 999^2 = 998001.$$

If we square 1, the smallest number of one figure, and 9, the largest number of one figure, the square consists of *twice as many figures as the number, or twice as many less one.* In the same way we find the principle true for numbers of two and three figures; hence, we infer that it is generally true.

PRINCIPLE II. *If we point off a number into periods of two figures each, beginning at the right, the number of full periods, together with the partial period at the left if there is one, will equal the number of figures in the square root.*

This is evident from Prin. I.

PRINCIPLE III. *The square of a decimal contains twice as many figures as the decimal itself.*

$$.1^2 = .01, \quad .9^2 = .81, \quad .01^2 = .0001, \quad .99^2 = .9801.$$

It is thus seen that the square of a decimal of one place occupies two places, and the square of a decimal of two places occupies four places; hence, we infer that the principle is generally true.

PRINCIPLE IV. *If we represent the units by u, tens by t, hundreds by h, and thousands by T, we will obtain the following formulas:*

$$(t+u)^2 = t^2 + 2\,tu + u^2.$$

$$(h+t+u)^2 = h^2 + 2\,ht + t^2 + 2(h+t)u + u^2, \text{ etc.}$$

1. Find the square root of 9025.

$$\begin{array}{rr|l}
 & t^2 + 2\,tu + u^2 = 90'25 & 90 \\
 & t^2 = 81\;00 & \underline{\;\;5} \\
\hline
(2\,t+u)u = \qquad\qquad & 9\;25 & 95, \textit{ Ans.} \\
2\,t = 2 \times 90 = 180 & & \\
+u \quad = \quad 5 & & \\
\hline
(2\,t+u)u = 185 = & 9\;25 &
\end{array}$$

Separating the number into periods of two figures each, we see that the root will contain two figures (Prin. II); hence, the root consists of *tens* and *units*, and the number consists of $t^2 + 2\,tu + u^2$.

The greatest number of *tens* whose square is contained in 9025 is 9 *tens*, or 90. Subtracting the square of 90 from 9025, we have 925 remaining, which equals $2\,tu + u^2$. Now, since $2\,tu$ is generally very much greater than u^2, 925 consists principally of $2\,tu$; hence, we can find the units by dividing 925 by $2\,t$, or 180; dividing 925 by 180 gives 5, the units' figure, and $(2\,t+u)\,u = (180+5)\,5$, or 925; subtracting, there is no remainder. Hence, the square root of 9025 is 95.

Therefore, to extract the square root of a number, we have the following

RULE. *Separate the number into periods of two figures each, beginning at units' place.*

Find the largest number whose square is contained in the left-hand period, write it as the first figure of the root, subtract its square from the number, and bring down the next period for a new dividend.

Divide the new dividend, omitting the last figure, by twice the root already found, and annex the quotient to the root and also to the divisor.

Multiply the complete divisor by the second term of the root, and subtract the product from the dividend.

Thus continue until all the periods are used.

EXAMPLES

2. Find the square root of 119,025.

In practice, the work is as follows:

$$\begin{array}{r|l|l} & 11'90'25 & 345 \\ & 9 & \\ \hline 64 & 290 & \\ & 256 & \\ \hline 685 & 3425 & \\ & 3425 & \\ \hline \end{array}$$

Find the square root of:

3. 3025.	7. 56,644.	11. 4,016,016.
4. 5184.	8. 321,489.	12. 11,594,025.
5. 7225.	9. 767,376.	13. 14,167,696.
6. 2209.	10. 944,784.	14. 250,050,969.

CUBE ROOT OF POLYNOMIALS

86. 1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{lr|l|l} & & a^3+3a^2b+3ab^2+b^3 & a+b, \textit{ Ans.} \\ & & a^3 & \\ \hline \text{Trial divisor,} & 3a^2 & 3a^2b+3ab^2+b^3 & \\ \text{1st correction,} & +3ab & & \\ \text{2d correction,} & +b^2 & & \\ \hline \text{Complete divisor,} & 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\ \hline \end{array}$$

From the method of cubing a polynomial (Art. 78), the first term, a^3, is the cube of the first term of the root; hence, the first term of the root is the cube root of a^3, which is a. Subtracting its cube from the polynomial, we have $3a^2b + 3ab^2 + b^3$ remaining.

The first term of the remainder is equal to *three times the square of the first term by the second;* hence, we can find the second term of the root by dividing $3a^2b$ by $3a^2$, which gives b. The first correction is equal to three times the product of the terms, or $3ab$, and the second correction is equal to the square of the second term, or b^2.

Adding, we find the complete divisor equals $3a^2+3ab+b^2$. Multiplying by the second term and subtracting, there is no remainder. Hence, the cube root is $a+b$.

2. Find the cube root of $a^6-3a^5+5a^3-3a-1$.

	$a^6-3a^5+5a^3-3a-1$	a^2-a-1,
	a^6	*Ans.*
$3a^4$	$-3a^5+5a^3$	
$-3a^3$		
$+a^2$		
$3a^4-3a^3+a^2$		
a^2	$-3a^5+3a^4-a^3$	
$3a^4-6a^3+3a^2$	$-3a^4+6a^3-3a-1$	
$-3a^2+3a$		
$+1$		
$3a^4-6a^3+\quad 3a+1$	$-3a^4+6a^3-3a-1$	

The first two terms are found as in Ex. 1. The second *trial divisor* may be found by taking three times the square of the root already found, thus, $3(a^2-a)^2 = 3a^4-6a^3+3a^2$; but a more convenient way is to repeat the second correction (a^2), and add it to the sum of the complete divisor and the two corrections, which gives $3a^4-6a^3+3a^2$.

Dividing the remainder by the new trial divisor, we have -1, the next term of the root. The first term is $3(a^2-a)(-1)$; the second correction is $(-1)^2$. Adding these corrections to the trial divisor, we have $3a^4-6a^3+3a+1$. Multiplying by -1 and subtracting, there is no remainder.

Hence, the cube root is a^2-a-1.

Therefore, to extract the cube root of a polynomial, we have the following

RULE. *Arrange the terms according to the powers of some letter.*

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the polynomial.

Divide the first term of the remainder by three times the square of the root already found, and write the quotient for the next term of the root.

Add to the trial divisor three times the product of the first and second terms of the root, and the square of the second term. Multiply the complete divisor by the second term of the root, and subtract the product from the remainder.

To find the second trial divisor, repeat the second correction, and add it to the sum of the preceding complete divisor and the first and second corrections.

Continue in this manner until all the terms of the root are found.

NOTE. The preceding method of cube root is the shortest and most convenient method now in use. The abbreviation consists in obtaining the successive trial divisors by using the previous work. It is applicable alike to polynomials and numbers, and the larger the problem the greater the abbreviation. The form given here should be carefully followed.

EXAMPLES

3. Find the cube root of:

$$\begin{array}{l|l}
 & m^6+6m^5-40m^3+96m-64 \,\big|\, m^2+2m-4, \\
 & m^6 \qquad\qquad\qquad\qquad\qquad\qquad \textit{Ans.} \\ \hline
3m^4 & 6m^5-40m^3 \\
\quad +6m^3 & \\
\qquad +4m^2 & \\ \hline
3m^4+6m^3+4m^2 & \\
\qquad 4m^2 & \\
 & 6m^5+12m^4+8m^3 \\ \hline
3m^4+12m^3+12m^2 & -12m^4-48m^3+96m-64 \\
\qquad -12m^2-24m & \\
\qquad\qquad +16 & \\ \hline
3m^4+12m^3 \quad -24m+16 & -12m^4-48m^3+96m-64 \\ \hline
\end{array}$$

Find the cube root of the following:

4. $m^3-3m^2n+3mn^2-n^3$.

5. $1-6m+12m^2-8m^3$.

6. $y^6+3y^5-5y^3+3y-1$.

7. $x^6+9x^5-135x^3+729x-729$.

8. $a^3-3a^2b+3ab^2-b^3$.

9. $m^6-3m^4n^2+3m^2n^4-n^6$.

10. $1-9x+39x^2-99x^3+156x^4-144x^5+64x^6$.

11. $y^6-9y^5+33y^4-63y^3+66y^2-36y+8$.

12. $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

13. $a^6-9a^5+33a^4-63a^3+66a^2-36a+8$.

14. $y^9-3y^8+6y^7-10y^6+12y^5-12y^4+10y^3-6y^2+3y-1$.

CUBE ROOT OF NUMBERS

87. PRINCIPLE I. *The cube of a number consists of three times as many figures as the number, or three times as many less one or two.*

$$1^3 = 1, \qquad 10^3 = 1000,$$

$$3^3 = 27, \qquad 30^3 = 27000,$$

$$9^3 = 729, \qquad 99^3 = 970299.$$

The cube of 1, the smallest number of one figure, is 1; the cube of 3 is 27, and the cube of 9, the largest number of one figure, is 729; hence, the cube of a number of one figure contains one, two, or three figures; that is, *three times as many, or three times as many less one or two.*

In the same way we find the principle true for numbers of two figures; hence, we infer that it is generally true.

PRINCIPLE II. *If a number is pointed off into periods of three figures each, beginning at the right, the number of full periods, together with the partial period on the left if there is one, will equal the number of figures in the root.*

This is evident from Prin. I.

PRINCIPLE III. *The cube of a decimal contains three times as many figures as the decimal itself.*

$$.1^3 = .001, \qquad .01^3 = .000001,$$

$$.9^3 = .729, \qquad .99^3 = .970299.$$

It is thus seen that the cube of a decimal of one figure contains three figures, and the cube of a decimal of two figures contains six figures; hence, we infer that the principle is generally true.

PRINCIPLE IV. *If we represent the units by* **u**, *the tens by* **t**, *and the hundreds by* **h**, *we will have the following formulas:*

$$(t+u)^3=t^3+3\,t^2u+3\,tu^2+u^3,$$

$$(h+t+u)^3=h^3+3\,h^2t+3\,ht^2+t^3+3\,(h+t)^2u+3\,(h+t)\,^2u+u^3.$$

1. Find the cube root of 300,763.

$$\begin{array}{l|r|l}
t^3+3\,t^2u+3\,tu^2+u^3= & 300'763 & 60 \\
t^3= \qquad\qquad\qquad 60^3= & 216\,000 & \underline{\;7\;} \\
\hline
(3\,t^2+3\,tu+u^2)\,u= & 84\,763 & 67,\ \textit{Ans.} \\
\quad 3\,t^2=3\times 60^2 \qquad = 10800 & & \\
+\,3\,tu=3\times 60\times 7\ = \ \ 1260 & & \\
\qquad\qquad +\,u^2=7^2= \qquad 49 & & \\
\hline
(3\,t^2+3\,tu+u^2)\,u=12109\times 7 & 84\,763 & \\
\hline
\end{array}$$

Separating the number into periods of three figures each, we find the root will contain two figures (Prin. II); hence the root will consist of tens and units, while the number consists of

$$t^3+3\,t^2u+3\,tu^2+u^3.$$

The greatest number of *tens* whose cube is contained in 300,763 is 6 *tens*, or 60. Subtracting the cube of 60 from 300,763, we have 84,763 remaining, which equals $3\,t^2u+3\,tu^2+u^3$.

Since $3\,t^2u$ is always greater than $3\,tu^2+u^3$, 84,763 consists principally of $3\,t^2u$; hence we can find the units' figure by dividing 84,763 by $3\,t^2$, or 10,800. Dividing 84,763 by 10,800 gives 7, the units' figure. The first correction is $3\,tu=3\times 60\times 7=1260$, and the second correction is $u^2=7^2=49$; taking the sum, we have 12,109. Multiplying by 7 and subtracting, there is no remainder. Hence the cube root is $60+7$, or 67.

2. Find the cube root of 95,256,152,263.

	95'256'152'263	4567, *Ans.*
	64	
48	31 256	
60		
25		
5425		
25	27 125	
6075	4 131 152	
810		
36		
615636		
36	3 693 816	
623808	437 336 263	
9576		
49		
62476609	437 336 263	

The first two figures of the root are found as in Ex. 1. The second trial divisor, 6075, may be found by taking three times 45^2; but a more convenient way is to repeat the second correction (25) and add it to the sum of the complete divisor and the two corrections, which gives 6075.

Dividing 4,131,152 by 6075, omitting the last two figures, we find the next figure of the root to be 6. Adding the first correction, $3 \times 6 \times 45 = 810$, and the second correction, $6^2 = 36$, we have the complete divisor 615,636. Multiplying by 6 and subtracting, we have 437,336,263.

The third trial divisor is found by repeating the second correction (36) and adding it to the last complete divisor and the first and second corrections, which gives 623,808. Dividing 437,336,263 by 623,808, omitting the last two figures, we find the next figure of the root to be 7. Adding the first and second corrections, multiplying by 7, and

subtracting, we have no remainder. Hence the cube root is 4567.

Therefore, to extract the cube root of a number, we have the following

RULE. *Separate the number into periods of three figures each, beginning with the units' place.*

Find the largest number whose cube is contained in the left-hand period, and write it as the first figure of the root; subtract its cube from the period, and bring down the next period for a new dividend.

Divide the new dividend, omitting the last two figures, by three times the square of the root already found, and write the quotient as the second figure of the root.

To the trial divisor add three times the product of the last figure of the root and the part of the root previously found, written one figure to the right, and the square of the last figure of the root, written two figures to the right, for the complete divisor.

Multiply the complete divisor by the figure of the root last found, and subtract the product from the remainder.

Thus continue until all the periods are used.

EXAMPLES

Find the cube root of:

3. 250,047.	7. 438,976.	11. 41,063,625.
4. 262,144.	8. 804,357.	12. 60,236,288.
5. 614,125.	9. 1,860,867.	13. 277,167,808.
6. 571,787.	10. 34,328,125.	14. 80,677,568,161.

VII. QUADRATIC EQUATIONS

88. A **Quadratic Equation** is an equation of the second degree.

Thus, $x^2 = b$, and $ax^2 + bx = c$.

89. There are two kinds of quadratic equations, (1) *Pure*, or *Incomplete* quadratics and (2) *Affected*, or *Complete* quadratics.

90. A **Pure Quadratic Equation** is one which contains only the second power of the unknown quantity; as, $x^2 = b$.

91. An **Affected Quadratic Equation** is one which contains both the first and the second powers of the unknown quantity; as, $ax^2 + bx = c$.

PURE QUADRATIC EQUATIONS

92. Examples.

1. Solve the equation $x^2 = 16$.

SOLUTION

$$x^2 = 16.$$

Extracting the square root, $x = \pm 4$.

2. Solve the equation $5x^2 - 10 = 115$.

Transposing, $5x^2 = 125$,

or $x^2 = 25$.

Extracting the square root, $x = \pm 5$.

NOTE. The sign $\pm$, read *plus* or *minus*, shows that the root is either $+$ or $-$, since $(+5) \times (+5) = 25$, and $(-5) \times (-5) = 25$.

Solve the equations:

3. $x^2 - 8 = 56.$

4. $x^2 + 15 = 136.$

5. $2x^2 - 5 = 283.$

6. $2x^2 - 4 = x^2 + 12.$

7. $(x-3)(x+3) = 40.$

8. $x^2 - a = b.$

9. $4x^2 + 15 = 415.$

10. $\dfrac{x^2}{a} + b = c.$

11. $cx^2 - b = 4.$

12. $ax^2 - b = cx^2 + d.$

13. $\dfrac{x^2}{3} + 4 = \dfrac{3x^2 - 1}{6}.$

14. $nx^2 - mx^2 = p.$

15. $(x+4)^2 + (x-2)^2 = 4x + 92.$

16. $(x+5)^2 + 4 = 10x + 38.$

17. $x^2 - 8 = 2(x^2 - 36).$

18. $\dfrac{x+5}{x-4} = \dfrac{2x+8}{x-5}.$

19. $(x+5)^2 + (x-5)(x-6) = -1x + 153.$

20. $(x+6)^2 + 32 = (x-8)^2 + x^2 + 28x.$

PROBLEMS

93. 1. The square of a number, plus 20 equals the square of half the number, plus 128. Find the number.

2. A lady being asked the number of pupils in her school, replied, "The square of half the number equals the square of one fourth of the number, increased by 75." How many pupils had she?

3. A farm containing 32 acres is 5 times as long as it is wide. Find its length and width.

4. An army of 3300 men, after sending out a detachment of 384 men, was formed into four equal squares. How many men were there in each side of a square?

5. The square of $\frac{2}{3}$ of a number is 72 less than $\frac{2}{3}$ of the square of the number. Find the number.

6. Five ninths of the square of a number is 8405. Find the number.

7. Fifty is $\frac{1}{4}$ of the difference between the square of twice a number and twice the square of a number. What is the number?

8. Three times the square of a number is 27 more than 9 times the square of half the number. What is the number?

9. Three fourths of the square of a number is 12 more than the square of $\frac{3}{4}$ of the number. What is the number?

10. One sixth of a number multiplied by one fifth of a number is 30. What is the number?

11. The difference between the square of twice a number and twice the square of a number is 8450. What is the number?

AFFECTED QUADRATICS

94. An **Affected Quadratic Equation** is solved by adding such a quantity to both members of the equation as will make the first member a perfect square. This process is called *completing the square.*

FIRST METHOD OF COMPLETING THE SQUARE

95. We found in Art. 36, Prin. I, that

$$(x+p)^2 = x^2 + 2px + p^2, \text{ and } (x-p)^2 = x^2 - 2px + p^2.$$

It is seen that $x^2 + 2px$ and $x^2 - 2px$ each lack p^2 of being a perfect square. If $x^2 + 2px$ or $x^2 - 2px$ is given, we can find p^2 by *dividing the second term, $2px$, by twice the square root of the first term, x^2,* thus

$$\sqrt{x^2} = x \text{ and } \frac{2px}{2x} = p.$$

1. Solve the equation $x^2 + 8x = 9$.

We have, $x^2 + 8x = 9.$

If we regard $x^2 + 8x$ as the first two terms of the square of a binomial, then the third term is found by dividing $8x$ by $2\sqrt{x^2}$ and squaring the result. $\frac{8x}{2x} = 4;\ 4^2 = 16.$

Completing the square, $x^2 + 8x + 16 = 25.$

Extracting the square root, $x + 4 = \pm 5.$

And, $x = -4 \pm 5,$

$x = -9, \text{ or } +1.$

2. Solve $x^2 + 10x = 24$.

We have $x^2 + 10x = 24.$

Completing the square, $x^2 + 10x + 25 = 49.$

Extracting the square root, $x + 5 = \pm 7.$

And, $x = -5 \pm 7,$

$x = 2, -12.$

3. Solve $x^2 - 16x = 36$.

We have $x^2 - 16x = 36.$

Completing the square, $x^2 - 16x + 64 = 100.$

Extracting the square, $x - 8 = \pm 10.$

And, $x = 8 \pm 10,$

$x = -2, 18.$

Note. **This method is used when the coefficient of x^2 is 1, and the coefficient of x is even.**

EXAMPLES

Solve the following equations:

4. $x^2+6x=40.$
5. $x^2-10x=75.$
6. $x^2-20x=21.$
7. $x^2+12x=108.$
8. $x^2+20x=69.$
9. $x^2-4x=96.$
10. $x^2+8x=180.$
11. $x^2+14x=15.$
12. $x^2-16x=36.$
13. $x^2-18x=40.$
14. $x^2+30x=99.$
15. $x^2-24x=25.$
16. $x^2-2x=399.$
17. $x^2+14x=-24.$

SECOND METHOD OF COMPLETING THE SQUARE

96. If a quadratic of the form

$$3x^2+5x=6$$

is solved by the preceding method, it will involve fractions, which will make the process long and difficult. The following method is used to avoid fractions:

1. Solve $2x^2+3x=5.$

To make the coefficient of x^2 a perfect square, we must multiply the equation by 2. To make the second term, $3x$, divisible by *twice the square root of the first term,* we must multiply the equation by 4; multiplying the equation by 8, we have

$$16x^2+24x=40.$$

Completing the square as in the first method, we have

$$16x^2+24x+9=49.$$

Extracting the square root,

$$4x+3=\pm 7,$$

and

$$x=\frac{\pm 7-3}{4}=1,\ -2\tfrac{1}{2}.$$

2. Solve $2x^2 - 3x = 2$.

We have $$2x^2 - 3x = 2.$$

Multiplying by 8, $$16x^2 - 24x = 16.$$

Completing the square,

$$16x^2 - 24x + 9 = 25.$$

Extracting the square root,

$$4x - 3 = \pm 5,$$

and $$x = \frac{\pm 5 + 3}{4} = 2,\ -\frac{1}{2}.$$

EXAMPLES

Solve the following equations:

3. $2x^2 + x = 21$.
4. $3x^2 - 2x = 21$.
5. $4x^2 + x = 18$.
6. $3x^2 - 4x = 55$.
7. $2x^2 + 7x = 22$.
8. $x^2 + x = 20$.
9. $2x^2 - 7x = -6$.
10. $3x^2 + 5x = 42$.
11. $5x^2 - 7x = 52$.
12. $3x^2 + 4x = 95$.
13. $3x^2 - 5x = 198$.
14. $6x^2 + 5x = 34$.
15. $3x^2 + 8x = 28$.
16. $ax^2 + bx = c$.

PROBLEMS

97. 1. The sum of two numbers is 14, and their product is 48. What are the numbers?

Let $$x = \text{one number}.$$

Then, $$14 - x = \text{the other number}.$$

By the condition, $x(14-x)=48.$

Expanding, $14x-x^2=48,$

and, $x^2-14x=-48.$

Whence, $x^2-14x+49=1.$

$$x-7=\pm 1.$$

$$x=7\pm 1=8, \text{ or } 6,$$

$$14-x=6, \text{ or } 8, \textit{ Ans.}$$

2. Find two numbers whose sum is 18, and whose product is 72.

3. Find two numbers whose difference is 7, and whose product is 260.

4. Divide 28 into two such parts that their product will be 75.

5. Find a number whose square increased by 6 times the number equals 72.

6. A man sold a cow for \$24, and gained as many per cent as the cow cost him dollars. Find the cost of the cow.

7. A man sold his horse for \$171, and gained as many per cent as the horse cost him dollars. What was the cost of the horse?

8. A sold goods for \$39, and gained as many per cent as the goods cost him dollars. Find the cost of the goods.

9. A rectangular farm is 3 times as long as it is wide. If it is made 10 rods wider and 20 rods longer, its area will be doubled. Find the area of the farm.

10. The length of a rectangular room exceeds its width by 8 feet, and the area is 560 square feet. What are the dimensions of the room?

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES

98. The **Degree** of an equation containing two unknown quantities is determined by the greatest sum of the exponents of the unknown quantities in any term.

Thus, $xy = 4$ is an equation of the second degree.

1. Solve the equations $\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$

SOLUTION

$$x + y = 7, \quad (1)$$
$$x^2 + y^2 = 25. \quad (2)$$

Square (1), $x^2 + 2xy + y^2 = 49.$ (3)

Subt. (2) from (3), $2xy = 24.$ (4)

Subt. (4) from (2), $x^2 - 2xy + y^2 = 1.$ (5)

Extract square root of (5), $x - y = \pm 1.$ (6)

Add (1) and (6), $2x = 8$, or 6.

$x = 4$, or 3, *Ans.*

and $y = 3$, or 4, *Ans.*

The pupil should always try to find an expression for $x + y$ and $x - y$. The value of x can then be found by addition, and the value of y by subtraction.

2. Solve $\begin{cases} x + y = 11, & (1) \\ x^2 - y^2 = 33. & (2) \end{cases}$

SOLUTION

Divide (2) by (1), $x - y = 3.$ (3)

Add (1) and (3), $2x = 14.$

$x = 7$, *Ans.*

and $y = 3$, *Ans.*

Solve the following equations:

3. $\begin{cases} x-y=4, \\ x^2-y^2=32. \end{cases}$

4. $\begin{cases} xy=12, \\ x+y=8. \end{cases}$

5. $\begin{cases} x^2+y^2=52, \\ x+y=10. \end{cases}$

6. $\begin{cases} x+y=12, \\ x^2-y^2=96. \end{cases}$

7. $\begin{cases} xy=18, \\ x^2+y^2=85. \end{cases}$

8. $\begin{cases} x+y=9, \\ x^3+y^3=351. \end{cases}$

9. $\begin{cases} x-y=2, \\ x^3-y^3=98. \end{cases}$

10. $\begin{cases} x+y=24, \\ x^2+y^2=416. \end{cases}$

11. $\begin{cases} x-y=16, \\ x^2-y^2=384. \end{cases}$

12. $\begin{cases} xy=18, \\ x^2+y^2=45. \end{cases}$

13. Solve $\begin{cases} 3x+y=9, & (1) \\ x^2+y^2=13. & (2) \end{cases}$

From (1), $y=9-3x.$ (3)

Substitute (y) in (2),

$$x^2+(9-3x)^2=13, \quad (4)$$

or $$x^2+81-54x+9x^2=13. \quad (5)$$

And $$10x^2-54x=-68, \quad (6)$$

or $$5x^2-27x=-34. \quad (7)$$

From (7), $x=2$, or $3\frac{2}{5}$, *Ans.*

14. $\begin{cases} 2x+y=11, \\ x^2-y^2=7. \end{cases}$

15. $\begin{cases} x+3y=7, \\ x^2+y^2=5. \end{cases}$

16. $\begin{cases} 2x+3y=5, \\ x^2+y^2=2. \end{cases}$

17. $\begin{cases} x-2y=2, \\ x^2-y^2=15. \end{cases}$

18. Solve $\begin{cases} x^3+y^3=9, & (1)\\ x^2y+xy^2=6. & (2)\end{cases}$

Mult. (2) by 3, $3x^2y+3xy^2=18.$ (3)

Add (3) to (1), $x^3+3x^2y+3xy^2+y^3=27.$ (4)

Extract the cube root of (4), $x+y=3.$ (5)

Divide (2) by (5), $xy=2.$ (6)

Square (5), $x^2+2xy+y^2=9.$ (7)

Subt. $4\times(6)$ from (7), $x^2-2xy+y^2=1.$ (8)

And $x-y=\pm 1.$ (9)

Add (5) and (9), $2x=4,\ 2,$

$x=2,\ 1,$ *Ans.*

And, $y=1,\ 2,$ *Ans.*

19. $\begin{cases} x^3+y^3=35,\\ x+y=\dfrac{30}{xy}.\end{cases}$

20. $\begin{cases} x^3+y^3=217,\\ x^2y+xy^2=42.\end{cases}$

21. $\begin{cases} x^3-y^3=208,\\ x^2y-xy^2=48.\end{cases}$

22. $\begin{cases} x^3+y^3=133,\\ x+y=\dfrac{70}{xy}.\end{cases}$

23. Solve $\begin{cases} x^2+xy+y^2=39, & (1)\\ x^4+x^2y^2+y^4=741. & (2)\end{cases}$

Divide (2) by (1), $x^2-xy+y^2=19.$ (3)

Subt. (3) from (1), $2xy=20.$ (4)

And, $xy=10.$ (5)

Add (5) to (1), $x^2+2xy+y^2=49.$ (6)

Subt. (5) from (3), $x^2-2xy+y^2=9.$ (7)

From (6), $x+y=\pm 7.$ (8)

From (7), $x-y=\pm 3.$ (9)

From (8) and (9), $x=\pm 5,$ *Ans.*

$y=\pm 2.$ *Ans.*

24. $\begin{cases} x^2+xy+y^2=31, \\ x^4+x^2y^2+y^4=651. \end{cases}$

26. $\begin{cases} x^2+xy+y^2=13, \\ x^4+x^2y^2+y^4=91. \end{cases}$

25. $\begin{cases} x^2+y^2+xy=28, \\ x^4+y^4+x^2y^2=336. \end{cases}$

27. $\begin{cases} x+\sqrt{xy}+y=7, \\ x^2+xy+y^2=21. \end{cases}$

MISCELLANEOUS EXAMPLES

28. $\begin{cases} xy=6, \\ x^4+y^4=97. \end{cases}$

32. $\begin{cases} (x+y)^2+4(x+y)=117. \\ x-y=7. \end{cases}$

29. $\begin{cases} x^2y+xy^2=20, \\ x^3+y^3=65. \end{cases}$

33. $\begin{cases} x^2+y^2=104, \\ x^4+y^4=10{,}016. \end{cases}$

30. $\begin{cases} x^2y=84+xy^2, \\ x^3-y^3=316. \end{cases}$

34. $\begin{cases} x^2+y^2+x+y=62, \\ xy=14. \end{cases}$

31. $\begin{cases} x^3+y^3=35, \\ x^2-xy+y^2=7. \end{cases}$

35. $\begin{cases} xy=12, \\ x^4+y^4=337. \end{cases}$

PROBLEMS

99. 1. The sum of two numbers is 14, and their product is 24. What are the numbers?

2. The sum of two numbers is 9, and the sum of their squares is 45. Find the numbers.

3. The sum of two numbers is 10, and the difference of their squares is 20. Find the numbers.

4. The sum of the squares of two numbers is 10, and the sum of their fourth powers is 82. Find the numbers.

5. The product of two numbers is 6 times their sum, and the sum of their squares is 325. Find the numbers.

6. A has two cubical boxes; the difference of their contents is 342 cubic inches, and the difference of their edges is 6 inches. What is the length of each box?

VIII. PROGRESSIONS

100. A **Series** is a succession of quantities which vary according to some fixed law.

Thus, 2, 4, 6, 8, 10, etc.

2, 4, 8, 16, 32, etc.

1, 4, 9, 16, 25, etc.

ARITHMETICAL PROGRESSION

101. An **Arithmetical Progression** is a series of quantities which vary by a common difference.

Thus, 1, 3, 5, 7, $\cdots$ is an *increasing* arithmetical progression, in which the common difference is 2.

6, 4, 2, 0, -2, -4, $\cdots$ is a *decreasing* arithmetical progression, in which the common difference is -2.

102. The **Quantities** considered in an arithmetical progression are *five*, any three of which being given the other two may be found. They are represented as follows:

(1) The first term, a. (3) The number of terms, n.

(2) The last term, l. (4) The common difference, d.

(5) The sum of the terms, s.

103. CASE I. **Given the first term, the common difference, and the number of terms, to find the last term.**

1. Given the first term a, the common difference d, and the number of terms n, to find the last term l.

$$\begin{aligned} &\text{The first term} &&= a, \\ &\text{the second term} &&= a + d, \\ &\text{the third term} &&= a + 2d, \\ &\text{the fourth term} &&= a + 3d, \\ &\text{the tenth term} &&= a + 9d, \\ &\text{the } n\text{th term} &&= a + (n-1)d. \end{aligned}$$

If we represent the nth term by l, we have

$$l = a + (n-1)d. \qquad \text{(I)}$$

2. Find the 20th term of the series 2, 4, 6, 8, etc.

In this example $a = 2$, $d = 2$, and $n = 20$. Substituting these values in
$$l = a + (n-1)d,$$
we have,
$$l = 2 + (20 - 1)2 = 40, \textit{ Ans.}$$

EXAMPLES

3. Find the 12th term of 1, 3, 5, 7, ⋯.
4. Find the 20th term of 3, 6, 9, ⋯.
5. Find the 30th term of 1, 4, 7, ⋯.
6. Find the 41st term of 2, 5, 8, ⋯.
7. Find the 15th term of 40, 46, 52, ⋯.
8. Find the nth term of 2, 4, 6, ⋯.
9. Find the nth term of 1, 3, 5, ⋯.
10. Find the nth term of 1, 4, 7, ⋯.

104. CASE II. **Given the first term, the last term, and the number of terms, to find the sum of the terms.**

1. Given the first term a, the last term l, and the number of terms n, to find the sum of the terms.

By adding the series we have

$$s = a + (a + d) + (a + 2d) + (a + 3d) \cdots l.$$

Inverting the series,

$$s = l + (l - d) + (l - 2d) + (l - 3d) \cdots a.$$

Adding these equations,

$$2s = (a + l) + (a + l) + (a + l)(a + l) \cdots (a + l).$$

It is seen that $(a + l)$ is taken as many times as there are terms, or n times; hence,

$$2s = n(a + l).$$

And $$s = \frac{n}{2}(a + l). \qquad \text{(II)}$$

EXAMPLES

2. Find the sum of 12 terms of the series $2+4+6+8 \cdots$.

In this series $a = 2$, $n = 12$, and $d = 2$.

Substituting these values in

$$l = a + (n - 1)d,$$

$$l = 2 + (12 - 1)2 = 24.$$

Substituting, $s = \frac{n}{2}(a + l)$,

$$s = \tfrac{12}{2}(2 + 24) = 156, \textit{ Ans.}$$

Find the sum :

3. Of 10 terms of $1 + 3 + 5 \cdots$.

4. Of 12 terms of $1 + 4 + 7 \cdots$.

5. Of 20 terms of $2 + 4 + 6 \cdots$.

6. Of 24 terms of $1 + 6 + 11 \cdots$.

7. Of 18 terms of $5 + 10 + 15 \cdots$.

8. Of n terms of $1 + 3 + 5 \cdots$.

9. Of n terms of $2 + 4 + 6 \cdots$.

10. Of n terms of $a + 2a + 3a \cdots$.

PROBLEMS

105. 1. A common clock strikes from 1 to 12; how often does it strike in 24 hours?

2. The Greenwich clock strikes from 1 to 24; how often does it strike in 24 hours?

3. If a body falls $16\frac{1}{12}$ feet the first second, three times as far the second, five times as far the third, and so on, how far will it fall in a minute?

4. If a stone dropped from a tower reaches the ground in 4 seconds, find the height of the tower. (See Ex. 3.)

5. The nth term of a series is $2n + 1$. Find the series.

6. The sum of n terms is n^2. Find the series.

7. The sum of n terms is $n^2 + n$. Find the series.

8. The sum of n terms is $3n^2 + 3n$. Find the series.

9. Find three numbers in arithmetical progression, such that their sum shall be 21, and their product 231.

NOTE. Represent the terms by $x - y$, x, and $x + y$.

10. Find three numbers in arithmetical progression, such that the sum of the squares of the first and second shall be 73, and the sum of the squares of the second and third shall be 233.

GEOMETRICAL PROGRESSION

106. A **Geometrical Progression** is a series of quantities which vary by a constant multiplier.

Thus, 2, 4, 8, 16, ⋯ is an *increasing* geometrical progression, in which the constant multiplier is 2.

81, 27, 9, 3, 1, ⋯ is a *decreasing* geometrical progression, in which the constant multiplier is $\frac{1}{3}$.

107. The **Quantities** considered in geometrical progression are *five*, any three of which being given the other two may be found. They are represented as follows:

(1) The first term, a. (3) The number of terms, n.
(2) The last term, l. (4) The ratio, r.
(5) The sum of the terms, s.

108. CASE I. **Given the first term, the ratio, and the number of terms, to find the last term.**

1. Given the first term a, the ratio r, and the number of terms n, to find the last term.

$$
\begin{aligned}
&\text{The first term} &&= a,\\
&\text{The second term} &&= ar,\\
&\text{The third term} &&= ar^2,\\
&\text{The fourth term} &&= ar^3,\\
&\text{The tenth term} &&= ar^9,\\
&\text{The } n\text{th term} &&= ar^{n-1}.
\end{aligned}
$$

If we represent the nth term by l, we have

$$\boldsymbol{l = ar^{n-1}} \qquad \text{(I)}$$

2. Find the 10th term of the series 2, 4, 8, 16, ⋯.

In this example $a=2$, $r=2$, and $n=10$. Substituting these values in

$$l=ar^{n-1},$$

$$l=2\times 2^{10-1},$$

$$l=2\times 2^{9},$$

$$l=2^{10}=1024,\ \textit{Ans.}$$

3. Find the 8th term of 1, 3, 9, ⋯.

4. Find the 12th term of 1, 2, 4, ⋯.

5. Find the 9th term of 3, 9, 27, ⋯.

6. Find the 11th term of 1, 4, 16, ⋯.

7. Find the nth term of 1, 3, 9, ⋯.

8. Find the nth term of 2, 4, 8, ⋯.

9. Find the nth term of a, $3a^2$, $9a^3$, ⋯.

10. Find the nth term of 1, -3, 9, -27, ⋯.

11. Find the nth term of 1, $\frac{1}{2}$, $\frac{1}{4}$, ⋯.

12. Find the nth term of 1, $\frac{1}{3}$, $\frac{1}{9}$, ⋯.

109. Case II. **Given the first term, the last term, and the number of terms, to find the sum of the terms.**

1. Given the first term a, the last term l, and the number of the terms n, to find the sum of the terms s.

By adding the series, we have

$$s=a+ar+ar^2+ar^3\cdots l. \tag{1}$$

Multiplying the series by r, we have

$$rs=ar+ar^2+ar^3\cdots+l+lr. \tag{2}$$

Subtracting (1) from (2), we have

$$(r-1)s = rl - a,$$

and $$s = \frac{rl-a}{r-1}. \tag{II}$$

By formula I $\quad l = ar^{n-1}.$

Then $\quad rl = ar^n.$

Substituting ar^n for rl in (II), we have

$$s = \frac{ar^n - a}{r-1}. \tag{III}$$

EXAMPLES

2. Find the sum of 8 terms of the series $2+4+8+16 \cdots$.

In this series, $\quad a = 2$, $r = 2$, and $n = 8$.

Substituting these values in

$$s = \frac{ar^n - a}{r-1},$$

we have $$s = \frac{2 \times 2^8 - 2}{2-1}.$$

Or, $$s = \frac{2^9 - 2}{1} = 510, \textit{ Ans.}$$

Find the sum:

3. Of 10 terms of the series $1 + 3 + 9 \cdots$.
4. Of 12 terms of the series $2 + 4 + 8 \cdots$.
5. Of 6 terms of the series $3 + 12 + 48 \cdots$.
6. Of 13 terms of the series $1 + 2 + 4 \cdots$.
7. Of 11 terms of the series $1 - 3 + 9 - 27 \cdots$.

8. Of 10 terms of the series $1 - 2 + 4 - 8 \cdots$.

9. Of n terms of the series $1 + 2 + 4 \cdots$.

10. Of n terms of the series $1 + 3 + 9 \cdots$.

11. Of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} \cdots$.

12. Of n terms of the series $1 + \frac{1}{3} + \frac{1}{9} \cdots$.

INFINITE SERIES

110. An **Infinite Series** is a series in which the number of terms is infinite; as, $1 + \frac{1}{2} + \frac{1}{4}$, etc.

The **Sum** of a decreasing geometrical series to infinity is the *limit* toward which the sum of the series approaches as the number of terms increase.

111. To find the sum of a decreasing geometrical series to infinity.

The value of s in formula (II) for a decreasing series is

$$s = \frac{a - rl}{1 - r}.$$

It is seen that in a decreasing geometrical series the last term becomes smaller as the number of terms increases, and when the number of terms is *infinitely* large the last term is infinitely small, and approaches the limit 0; therefore rl approaches the limit 0 and may be disregarded.

Therefore, the limit of $\frac{a - rl}{1 - r} = \frac{a}{1 - r}$.

Hence the sum of a decreasing geometrical series to infinity is

$$s = \frac{a}{1 - r}. \qquad \text{(IV)}$$

EXAMPLES

Find the sum of the infinite series:

1. $1 + \frac{1}{2} + \frac{1}{4} \cdots$.
2. $1 + \frac{1}{3} + \frac{1}{9} \cdots$.
3. $1 + \frac{1}{4} + \frac{1}{16} \cdots$.
4. $1 + \frac{1}{5} + \frac{1}{25} \cdots$.
5. $1 + \frac{1}{6} + \frac{1}{36} \cdots$.
6. $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} \cdots$.
7. $\frac{1}{5} + \frac{2}{25} + \frac{4}{125} \cdots$.
8. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \cdots$.
9. $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} \cdots$.
10. $\frac{2}{5} - \frac{4}{25} + \frac{8}{125} \cdots$.

PROBLEMS

112. 1. The nth term is 5^{n-1}. Find the series.

2. The nth term is a^{2n}. Find the series.

3. The sum of n terms is $2^n - 1$. Find the series.

4. The sum of n terms is $\frac{1}{2}\left(\frac{3^n - 1}{3^{n-1}}\right)$. Find the series.

5. A man agreed to labor at the rate of \$1 for January, \$2 for February, \$4 for March, and so on to the end of the year. How much did he receive?

6. A agreed to labor 20 days on condition that he should receive 1 cent for the first day, 2 cents for the second day, 4 cents for the third day, and so on. How much did he receive?

GENERAL REVIEW

I. ALGEBRAIC NOTATION (pp. 9–19)

113. Find the value of the following expressions when $x=10$ and $y=4$:

1. $\dfrac{x^2-y^2}{x+y}$.
3. $\dfrac{x^3-y^3}{x-y}$.
5. $\dfrac{x^4-y^4}{x-y}$.
2. $\dfrac{x^2-y^2}{x-y}$.
4. $\dfrac{x^3+y^3}{x+y}$.
6. $\dfrac{x^4-y^4}{x^2-y^2}$.

7. Show that $(x+y)(x-y)=x^2-y^2$.

8. Show that $(x+y)^2=x^2+2xy+y^2$.

9. Show that $(x-y)^2=x^2-2xy+y^2$.

10. Show that $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

11. Show that $(x-y)^3=x^3-3x^2y+3xy^2-y^3$.

12. Show that $\dfrac{x^5-1}{x-1}=x^4+x^3+x^2+x+1$.

13. Find the value of $\sqrt{y^2+\sqrt{x+y+67}}$.

14. Find the value of $\sqrt{x^2+y^2+\sqrt{x+15}}$.

II. FUNDAMENTAL PROCESSES (pp. 20–48)

15. Add $x^2+2xy+y^2$, $x^2-2xy+y^2$, x^2+xy+y^2, x^2-xy+y^2, $x^2+4xy-y^2$, $x^2-5xy-y^2$, $-9x^2-6xy-9y^2$, $16x^2-8xy+16y^2$, $6x^2-12xy+6y^2$, $10x^2-10xy+10y^2$, and $4x^2-xy-2y^2$.

16. Add $4\sqrt{a+b}+bc-a$, $3a-5\sqrt{a+b}-4bc$, $8\sqrt{a+b}-bc$, $5a+10bc$, $4a+3\sqrt{a+b}-7bc$, $4a-6\sqrt{a+b}-bc$, $9\sqrt{a+b}-a+3bc$, $5\sqrt{a+b}-8a+5bc$, and $9bc-8\sqrt{a+b}-7a$.

17. From the sum of $m^2+3mn-n^2$ and $m^2-4mn+n^2$ take $m^2+5mn-n^2$.

18. From $a^6+5a^5+4a^4+3a^3+2a^2+a+1$ take $a^6+5a^5-4a^4-3a^3+2a^2-a+1$.

19. Find the value of

$$8n-[x-y+\{n+x-(b-n)+y\}-n-y].$$

20. Find the value of

$$x-[-4a-\{-6b-(-8d+\overline{6b-a+d})\}].$$

21. Multiply $a^{m+n+1}+b^{m+n-1}$ by $a^{m+n+1}-b^{m+n-1}$.

22. Multiply $m^6-m^5+m^4-m^3+m^2-m+1$ by $m+1$.

23. Multiply a^2+ab+b^2, a^2-ab+b^2, $a^4-a^2b^2+b^4$, and $a^8-a^4b^4+b^8$.

24. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

25. Divide x^n+y^n by $x+y$ to seven terms.

26. Divide x^n-y^n by $x-y$ to seven terms.

27. Solve $8x-6+5x-9=4x+30$.

28. Divide x^7-y^{14} by $x-y^2$.

29. Expand $(m-n)(m+n)(m^2+n^2)(m^4+n^4)$.

30. From $18m^2-6mn-5n^2$ take $12m^2+10mn-6n^2$.

31. Expand $(x^2+y^n)(x^2-y^n)$.

32. Multiply x^n-xy+y^n by x^n+xy+y^n.

33. Expand $(3x-2)(3x+2)(9x^2+4)(81x^4+16)$.

34. Divide $a^8+12a^6x+54a^4x^2+108a^2x^3+81x^4$ by $a^4+9x^2+6a^2x$.

35. Expand $(x^2+y^3)^2$ and $(2a^2-3x^3)^2$.

36. Expand $(a^2+b^2)(a^4-a^2b^2+b^4)$.

37. Find the value of

$$-\{a+b-[a-(a-\overline{b-c})+b]-b+c\}.$$

38. Find the value of

$$2x-\{3a+(2y-\overline{4a-6x})\}-\{3a+(5x-\overline{8a-5y})\}.$$

39. Divide x^7+y^7 by $x+y$ and x^8-y^8 by $x-y$.

40. Multiply $(a-3)(a-3)$ by $(a+3)(a+3)$.

41. Divide $a^{24}+b^{24}$ by a^8+b^8.

III. COMPOSITION AND FACTORING (pp. 49–64)

42. Expand $(2x-3y)^2$ and $(nx+my)^2$.

43. Expand $(2ab+3c)(2ab-3c)$.

44. Expand $(y+7)(y-3)$ and $(a-c)(a+b)$.

45. Factor $(x+y)^2-(x-y)^2$ and $x^4+x^2y^2+y^4$.

46. Factor $a^{12}-b^{12}$ and $a^{24}-b^{24}$.

47. Factor $x^2-26x-120$ and $x^{16}-y^{16}$.

48. Factor $x^2+53x+360$ and a^6-37a^3-360.

49. Find the H. C. D. of $x^2-(a+b+c)^2$ and $(x-a)^2-(b+c)^2$.

50. Find the H. C. D. of (x^6+y^6), $x^4-x^2y^2+y^4$, and $x^8+x^4y^4+y^8$.

51. Find the L. C. M. of a^2-2a+1, a^2-6a+5, and a^2-4a+3.

52. Find the L. C. M. of x^3-y^3, x^2+xy+y^2, $x-y$, and x^2-y^2.

53. Find the L. C. M. of $a^2+4ab-21b^2$, $a^2-2ab-3b^2$, and $a^2-6ab-7b^2$.

54. Factor $m^2 - 2xy + n^2 - x^2 + 2mn - y^2$.

55. Factor $x^2 - 1 + y^2 + 2xy$.

56. Factor $(m^2 + n^2)^2 - (a^2 - b^2)^2$.

57. Factor $a^2 - y^2 - 2ab + b^2$.

58. Factor $1 - m^2 - n^2 - 2mn$.

59. Factor $a^2 - 23a + 60$, and $x^2 - 36x + 99$.

60. Factor $a^2 + 6a - 91$, and $x^2 + 42a + 441$.

61. Factor $m^2 - 10m + 24$, and $x^2 - 26x + 105$.

62. Find the H. C. D. of $a^4 - 16$, $a^2 - a - 6$, and $(a^2 - 4)^2$.

63. Find the H. C. D. of $m^3 - m$, $m^3 + 9m^2 - 10m$, and $m^6 - m$.

64. Find the H. C. D. of $x^2 - 1$, $x^3 + 1$, and $x^2 + 2x + 1$.

65. Find the L. C. M. of $x^2 - y^2 - z^2 + 2yz$ and $x^2 - y^2 + z^2 + 2xz$.

IV. FRACTIONS (pp. 65–85)

66. Reduce $\dfrac{x^6 + b^9}{x^4 - b^6}$ to its lowest terms.

67. Reduce $\dfrac{x^2 + 11x - 60}{x^2 - 18x + 56}$ to its lowest terms.

68. Reduce $\dfrac{x^{10} + y^{10}}{x^6 + y^6}$ to its lowest terms.

69. Reduce $a^2 + 2a + 1 - \dfrac{4a^2 + 6a}{a + 2}$ to a fraction.

70. Reduce $\dfrac{2x - 4}{2x + 6}$, $\dfrac{5x + 20}{15x + 45}$, $\dfrac{3x - 15}{12x + 36}$ to a common denominator.

71. Add $\dfrac{x}{(y-x)(y-z)}$ and $\dfrac{y}{(x-y)(x-z)}$.

72. Subtract $\dfrac{b}{(a-b)(b-c)}$ from $\dfrac{a}{(a-b)(c-b)}$.

73. Multiply $\dfrac{x^2+y^2-z^2+2\,xy}{x^2-y^2-z^2+2\,yz}$ by $\dfrac{x-y+z}{x-y-z}$.

74. Divide $\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)$ by $\left(\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}\right)$.

75. Expand $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^2$ and $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)^2$.

76. Factor $m^6+\dfrac{n^6}{x^6}$ and $\dfrac{x^3}{y^3}-n^3$.

77. Factor $m^2+2+\dfrac{1}{m^2}$ and $x^2+2+\dfrac{1}{x^2}$.

78. Factor $a^2-\left(\dfrac{a^2+b^2-c^2}{2\,b}\right)^2$.

79. Multiply $\dfrac{x^2+2x-63}{x^2+4x-21}$, $\dfrac{x^2-10x+21}{x^2+4x-45}$, and $\dfrac{x^2+2x-35}{x^2-14x+49}$.

80. Simplify $\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)\div\left(\dfrac{a^2}{a^2+b^2}+\dfrac{b^2}{a^2-b^2}\right)$.

81. Simplify $\dfrac{(a+2)^2-9}{(a+3)^2-16}\div\dfrac{(a+1)^2-16}{(a+5)^2-4}$.

82. Simplify $\dfrac{\dfrac{2}{3-a}+6-a}{\dfrac{24}{3-a}-7-a}$.

83. Simplify $\dfrac{\dfrac{1}{(a+b)(a-c)}+\dfrac{1}{(a-b)(a+c)}}{\dfrac{1}{(a+b)(a+c)}+\dfrac{1}{(a-b)(a-c)}}$.

V. SIMULTANEOUS EQUATIONS (pp. 86–94)

Solve the following equations:

84. $\begin{cases} 4x - 5y = 5, \\ 10x + 7y = 71. \end{cases}$

85. $\begin{cases} x + \dfrac{y-2}{3} = 8, \\ \dfrac{x+3}{3} - y = 1. \end{cases}$

86. $\begin{cases} \dfrac{2x+y}{x-3y} = \dfrac{29}{4}, \\ \dfrac{x+4}{2} - \dfrac{x}{2} - \dfrac{y-1}{4} + \dfrac{7}{4} = 3\frac{1}{4}. \end{cases}$

87. $\begin{cases} \dfrac{x}{m} - \dfrac{y}{n} = a, \\ \dfrac{x}{n} + \dfrac{y}{m} = b. \end{cases}$

89. $\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} = 1, \\ \dfrac{n}{x} + \dfrac{m}{y} = 1. \end{cases}$

88. $\begin{cases} ax + by = c, \\ bx + ay = d. \end{cases}$

90. $\begin{cases} .5x + .7y = m, \\ .05x - .9y = n. \end{cases}$

91. If the length of a rectangle is increased 5 feet and the width 3 feet, the area will be increased 270 square feet; but if the length is diminished 3 feet and the width 5 feet, the area will be diminished 266 square feet. What are the dimensions of the rectangle?

92. A, B, and C can do a piece of work in 20 days; B, C, and D in 24 days; A, C, and D in 30 days; and A, B, and D in 36 days. How long will it take each to do it?

93. A boy bought a apples and b oranges for n cents, and at the same rate b apples and a oranges for m cents. What was the price of each?

94. A and B play marbles: A loses to B as many as B has, then B loses to A as many as A now has, after which it is found that each has 24 marbles. How many had each at first?

95. A and B together had \$ 2200: A lost $\frac{1}{5}$ of his, and B lost $\frac{1}{3}$ of his; when it was found that each had the same sum of money. How much had each at first?

96. In a political convention a resolution was passed by a majority of 30 votes; if $\frac{2}{5}$ of those who voted for it had voted against it, the resolution would have been lost by 42 votes. How many voted?

97. A certain sum of money at simple interest amounted in m years to \$ a, and in n years to \$ b. Find the sum of money and the rate of interest.

98. A boy went to a store and spent \$ $\frac{1}{2}$ more than $\frac{1}{2}$ of his money; at a second store he spent \$ $\frac{1}{2}$ more than $\frac{1}{2}$ of what he had remaining; and at a third store he spent \$ $\frac{1}{2}$ more than $\frac{1}{2}$ of what he had yet remaining, and then found that he had \$ 5 left. How much had he at first?

99. In a mile race M gives N a start of 20 rods, and beats him 40 seconds; in the second heat M gives N a start of 48 seconds, and beats him 10 rods. What is the rate of each per hour?

VI. INVOLUTION AND EVOLUTION (pp. 95–113)

Square:

100. $a+l+g+e+b+v+o$, and $2a+3b+6c+d$.

101. $r+e+p+u+b+l+i+c$, and $1+x+x^2+x^3$.

102. $l-m-p-q-r+s$, and $A+Bx+Cx^2+Dx^3$.

Cube:

103. $m+x-y$, and $a-b-c-d$.

104. $m-n-p+y$, and $x+y-z-w$.

Expand:

105. $(a+b)^8$. **108.** $(2a+b)^4$. **111.** $(5a-6b)^5$.

106. $(x-1)^6$. **109.** $(3x-2y)^5$. **112.** $(a+b)^n$.

107. $(1-x)^6$. **110.** $(2x^2-3y^2)^4$. **113.** $(a-b)^n$.

Find the square root of the following:

114. $4a^6+5a^4+12a^5-5a^2-10a^3+2a+1$.

115. $a^2-4ab+4b^2+6ac-12bc+9c^2-8ad+16bd-24cd+16d^2$.

116. $a^2-6ab+9b^2+8ax-24bx+16x^2-4ay+12by-16xy+4y^2$.

Find the cube root of the following:

117. $27m^6+171m^4-54m^5-188m^3-150m+125+285m^2$.

118. $a^3-3a^2+9a-13+\frac{18}{a}-\frac{12}{a^2}+\frac{8}{a^3}$.

119. $a^{6n}-6a^{5n}+21a^{4n}-44a^{3n}+63a^{2n}-54a^n+27$.

120. $\sqrt[3]{1{,}371{,}330{,}631}$.

121. $\sqrt[3]{706{,}157{,}817{,}625}$.

VII. QUADRATIC EQUATIONS (pp. 114–124)

Solve the following equations:

122. $(x+2)^3+(x-2)^3=56x$.

123. $(x+3)^2+(x-5)(x+4)=61+5x$.

124. $(x-2)(x+3)+(x+4)(x-2)=274+3x$.

125. $\frac{x+5}{x-5}+\frac{x-5}{x+5}=6\frac{1}{6}$.

126. $\frac{x^2}{3}+4=\frac{3x^2-1}{6}$.

127. $x^2-14x=51$.

128. $x^2-20x=300$.

129. $x^2+30x=-29$.

130. $x^2-10x=-9$.

131. $x^2+4x=1$.

132. $(x+3)^2+2ax=9$.

133. $ax^2+4a^2x+ab-5a^2=0$.

134. $x^2-120x=1584$.

135. $x^2+100x=204$.

136. $y^2-60y=-500$.

137. $x^2-nx=m$.

138. $x^2+2(n+1)x=-4n$.

139. $x^2+2(m-n)x=4mn$.

140. $x^2-2(a+b)x=3a^2-b^2-2ab$.

141. Three times the square of a number increased by 5 exceeds five times the number by 177. What is the number?

142. A flower bed is 8 feet wide and 10 feet long. How wide must a walk be made around the outside of it to contain as many square feet as the bed?

143. How much are buttons per dozen if 6 dozen cost as many cents as the number of buttons you can buy for 8 cents?

144. $\begin{cases} \frac{1}{x}+\frac{1}{y}=5, \\ \frac{1}{x^2}+\frac{1}{y^2}=13. \end{cases}$

145. $\begin{cases} 2x-y=10, \\ xy=12. \end{cases}$

146. $\begin{cases} 8x^3-y^3=37, \\ 2x-y=1. \end{cases}$

147. $\begin{cases} xy=20, \\ x^2+y^2=41. \end{cases}$

148. $\begin{cases} x+y+\sqrt{x+y}=12, \\ x-y+\sqrt{x-y}=2. \end{cases}$

149. $\begin{cases} \frac{x^2}{y^2}+6\frac{x}{y}=16, \\ x+y=6. \end{cases}$

150. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 4, \\ \dfrac{1}{x^2} - \dfrac{1}{y^2} = 24. \end{cases}$

151. $\begin{cases} 2x + y = 9, \\ 3x^2 + 2y^2 = 62. \end{cases}$

152. $\begin{cases} 2x^2 + xy = 24, \\ x + 2y = 7. \end{cases}$

153. $\begin{cases} x^2 + y^2 + x + y = 62, \\ xy = 14. \end{cases}$

154. $\begin{cases} (x+y)^2 + 4(x+y) = 117, \\ x - y = 7. \end{cases}$

155. $\begin{cases} x^2 + y^2 = 5, \\ x^6 + y^6 = 65. \end{cases}$

156. The sum of two numbers is 12, and the sum of their cubes is 468. What are the numbers?

157. The sum of two numbers multiplied by their difference is 60, and their product is 16. What are the numbers?

158. The distance around a rectangular field is 60 rods, and its area is 200 square rods. Find its dimensions.

159. A and B have each a square field. The combined area is 2500 square rods, and 280 rods of fence are required to inclose both. What is the value of each, at $80 per acre?

160. The fore wheel of a carriage makes 10 revolutions more than the hind wheel in going 120 yards; but if the circumference of each is increased 1 yard, the fore wheel will make only 6 more revolutions than the hind wheel in going the same distance. What is the circumference of each wheel?

161. The sum of the squares of two numbers is 25, and the sum of the products obtained by multiplying each number by the cube of the other is 300. Find the numbers.

VIII. PROGRESSIONS (pp. 125–133)

162. Given $\frac{x-a+d}{d}=\frac{2s}{a+x}$ to find x.

163. Given $a+(x-1)d=\frac{2s-ax}{x}$ to find x.

164. Given $\frac{l-x+d}{d}=\frac{2s}{x+l}$ to find x.

165. Given $\frac{2s-lx}{x}=l-(x-1)d$ to find x.

166. Given $l=36$, $n=12$, and $s=234$ of an arithmetical progression to find the first term.

167. Given $a=3$, $d=5$, and $s=255$ of an arithmetical progression to find the last term.

ANSWERS

Page 9. — Ex. **3.** $7a$; $12a$. **4.** $9a$; $15b$. **5.** $18n$; $18x$. **6.** $13n$; 26; 39.

Page 10. — Ex. **7.** 18; 45. **8.** 48; 120. **9.** 34; 62. **10.** 51; 120.

3. $6a$; $7y$. **4.** $8x$; $8n$. **5.** $11x$; 0. **6.** 48; 144. **7.** 50; 125. **8.** 204; 340. **9.** 136. **10.** 80.

Page 11. — Ex. **1.** xy; mn. **2.** abd; mnp. **3.** 20. **4.** 60. **5.** 420. **6.** 16; 216. **7.** 25; 125; 625. **8.** a^6. **9.** 24. **10.** 488; $-\frac{11}{8}$.

Page 12. — Ex. **11.** 36. **12.** 72. **13.** x^4 by 69; $4x$ by $\frac{107}{81}$. **14.** $x+x+x+x+x+x+x$; $xxxxxxx$. **15.** 9; 27; 12; 81. —— **1.** 4. **2.** 16. **3.** 40. **4.** 32. **5.** 27; 12. **6.** 4; 6; 8; a; x; n. **7.** 2; 3; 4; x; y; n.

Page 13. — Ex. **8.** 11; 5. **9.** 10. **10.** 6. **11.** 6. **12.** x; xy. **13.** n; xyz. **14.** $x+y$; $x-y$; xy; $x \div y$; x^2; y^2; $\sqrt{x}$; $\sqrt{y}$; $\sqrt[3]{x}$; $\sqrt[3]{y}$.

Page 14. — Ex. **15.** 80. **16.** 0. **17.** 20. **18.** 180. **19.** 200. **20.** 288. **21.** 12. **22.** 76.

Page 15. — Ex. **2.** 10. **3.** 10. **4.** 36. **5.** 273. **6.** 4. **7.** 4161.

Page 16. — Ex. **2.** $y=3$. **3.** $x=9$. **4.** $x=10$. **5.** $x=7$. **6.** $x=20$. **7.** $x=10$. **8.** $x=8$. **9.** $y=16$. **10.** $y=5$. **11.** $x=5$.

Page 17. — Ex. **2.** $x=2$. **3.** $x=2$. **4.** $x=6$. **5.** $x=4$. **6.** $x=3$. **7.** $x=5$. **8.** $x=5$. **9.** $x=5$. **10.** $x=8$. **11.** $x=4$. **12.** $x=2$. **13.** $x=12$. **14.** $x=5$. **15.** $x=2$. **16.** $x=5$. **17.** $x=6$. —— **2.** 12; 96.

Page 18. — Ex. **3.** A, 360 acres; B, 90 acres. **4.** A, 84 years; B, 14 years. **5.** M, 1616 cattle; N, 202 cattle. **6.** P, \$70,500; Q, \$49,500. **7.** R, \$60,000; S, \$50,000; T, \$40,000. **8.** A, \$120,000; B, \$60,000; C, \$30,000. **9.** 150; 30. **10.** Horse, \$200; wagon, \$40. **11.** 240. **12.** Cow, \$20. —— **1.** $n-b$; $n+c$. **2.** wx; $6wy$. **3.** $x+1$; $x-1$.

Page 19. — Ex. **4.** $10y+x$. **5.** $100h+10t+u$; $100u+10t+h$. **6.** $100a+25b+10c$. **7.** $2x+2y$. **8.** $6x^2$ sq. ft.; $\frac{x^3}{27}$ cu. yd. **9.** $\frac{abc}{128}$ cords. **10.** $\frac{1}{a}$; $\frac{n}{a}$. **11.** $\frac{1}{x}+\frac{1}{y}$; $\frac{n}{x}+\frac{n}{y}$. **12.** nx. **13.** x^n. **14.** $\frac{ab}{160}$ acres.

Page 20. — Ex. **2.** $22x$. **3.** $29y$. **4.** $20ab$. **5.** $-23xy^2$. **6.** $-23am^2$. **7.** $43x$. **8.** $34a$. **9.** $27xy$. **10.** $-25a$.

Page 21.—Ex. **11.** $-29x^2$. **12.** $-29a$. **13.** $57y$. **14.** $54ay$. **15.** $113am$. **16.** $60mn$. **17.** $255x^2y^2$. **18.** $69xy^2$. **19.** $-137n^2$. **20.** $-80abc$.

Page 22.—Ex. **23.** 9. **24.** -6. **25.** $11a$. **26.** $-10n$. **27.** $4abc$. **28.** $18x^2y^2$. **29.** $-16mn$. **30.** $19ab^2$. **31.** $3a$. **32.** $6ab$. **33.** $14n^2$. **34.** $30ax$. **35.** $-14m^2x$. **36.** $-43x^2y^2$.

Page 23.—Ex. **3.** $6a-6b$. **4.** $-5ab+9xy$. **5.** $6a-14b-10c$. **6.** $34a-23b$. **7.** $18a-x-23y$. **8.** $20x^3+14x^2+8x$. **9.** $a^3-2a^2b-7ab^2-7b^3$. **10.** $10x^2-15xy-4y^2$. **11.** $2m^4+12m^2n^2+2n^4$. **12.** $-7m^2+mn+13n^2$. **13.** 0.

Page 24.—Ex. **15.** 0. **16.** $14(a-b)$. **17.** $-19ax+23by$.

Page 25.—Ex. **4.** $5x$. **5.** $-5x$. **6.** $7ax^2$. **7.** $-7ax^2$. **8.** $8m-8n$. **9.** $2a-b-c$. **10.** $9b-2d+2a^2$. **11.** ab. **12.** $-x^2y^2$. **13.** $2x^3-2$. **14.** $m^2-2mn+n^2$. **15.** a^3+b^3. **16.** $-x^2-4x^4$. **17.** $-c^3$. **18.** $2(a+b)-6(c-d)$.

Page 26.—Ex. **3.** $a-c$. **4.** $a+c$. **5.** $-a-c$. **6.** $-a+c$. **7.** $23a$. **8.** $23a$. **9.** $5a$. **10.** $-23a$. **11.** $-23a$. **12.** $-5a$. **13.** $5a$. **14.** $-5a$. **15.** $4ab$. **16.** $-2b+2c$. **17.** $2ab$.

Page 27.—Ex. **18.** $2+y+xy+2y^2$. **19.** $2xy+2y^2$. **20.** $6a^2b+2b^3$. **21.** $2x^4+2x^2+2$. **22.** $5x^3y+5x^2y^2+5xy^3$. **23.** $6x^4y+9x^3y^2+11x^2y^3+4xy^4+2y^4$. **24.** $-2x^m-2x^ny^m+8y^n$. **25.** $-6x^2+11y^3+16z^2$. **26.** $3(a-b)+15(c+d)$. **27.** $7m^2-14mn+1$.

Page 28.—Ex. **1.** $2x^2-xy+y^2$. **2.** $ab+b^2$. **3.** 0. **4.** $2xy$. **5.** $2x^2-3y^2$. **6.** $4ab$. **7.** n^2-mn. **8.** $2a^2-2ab$. **9.** $(x-y)+(a-b)$. **10.** $(m+n)+(x-y)$. **11.** $(4-x)+(y-a)$. **12.** $(a-x)+(m+n)$. **13.** $(a-b)+(m+n)+(x-y)$. **14.** $(c+d)+(m-n)+(x+z)$. **15.** $(2-x)+(a-b)+(m-n)$. **16.** $(a-b)+(c-d)+(m-n)$.

Page 29.—Ex. **17.** $a+b$. **18.** y. **19.** $2x-y$. **20.** $-n$. **21.** $2b$. **22.** $-2y$. **23.** $-m+n$. **24.** $a-5b$. **25.** $2x^2$. **26.** 0. **27.** 0. **28.** $(m-n)-(p-q)-(d-x)+(y-z)$. **29.** $-(a-b)-(m+n)+(2-3y)-(2z+m^2)$. **30.** $(x-2y)+(z-4)-(m+n)-(p+q)$. **32.** $(a-m)x+(b-n)y-(c+p)z$. **33.** $(2+a)x-(3+b)y-(4-c)z$. **34.** $(4m-3n)x-(3a+5c)y+(5b-8m)z$.

Page 30.—Ex. **36.** $a+2b-c$; $a-b-c+d$. **37.** $a-2b+c$; $-n+b$. **38.** b. **39.** $x-5a+6b+8d$. **40.** $x+5a-12b+7d$.

Page 33.—Ex. **3.** -8. **4.** -10. **5.** 12. **6.** $-ax$. **7.** $-ax$. **8.** ax. **9.** a^5. **10.** x^7. **11.** m^6. **12.** $-a^{12}$. **13.** $-x^{11}$. **14.** m^9. **15.** x^{m+n}. **16.** a^{3n}. **17.** x^{7n}. **18.** x^{2m}. **19.** a^{2n}. **20.** x^{m+n+5}. **21.** $-a^{2n+1}$. **22.** $-a^{n+2}$. **23.** n^{7n-1}. **24.** y^{2n+2}. **25.** x^7. **26.** a^4. **27.** $-a^9$. **28.** a^{10}. **29.** $-x^{15}$. **30.** y^{6n}. **31.** $6x^3-54x$. **32.** $3a^2bc+3ab^2c-3abc^2$. **33.** $4m^3n-2mn^3$. **34.** $3x^3y-3xy^3$. **35.** $4a^3bc+4ab^3c-4abc^3$. **36.** $3x^3y+3xy^3+3x^2y^2$. **37.** $5x^3y-10x^2y^2+5xy^3$. **38.** $-3a^3b+9a^2b^2-3ab^3$. **39.** $3x^4+3x^3+3x^2+3x$. **40.** $3a^5b-9a^3b^3+3ab^5$. **41.** $-2a^4b+6a^3b^2-2ab^4$. **42.** $6m^3n+6m^2n^2+6mn^3$. **43.** $4a^5+4a^4-4a^2$. **44.** $4a^{2n}b^n-4a^{n+2}b^{n+2}+4a^nb^{2n}$.

Page 35. — Ex. **7.** $x^2 + 10x + 25$. **8.** $x^2 - 10x + 25$. **9.** $x^2 - 25$. **10.** $x^2 + 10x + 21$. **11.** $x^2 - 10x + 21$. **12.** $x^2 - 11x + 24$. **13.** $x^2 + 2xy + y^2$. **14.** $x^2 - 2xy + y^2$. **15.** $x^2 - y^2$. **16.** $x^4 - y^4$. **17.** $6x^5 + 4x^2y^2 - 9x^3y^3 - 6y^5$. **18.** $m^6 - n^4$. **19.** $a^6 + b^6$. **20.** $m^8 + m^4 + 1$. **21.** $a^4 + a^2 + 1$. **22.** $x^4 - 1$. **23.** $x^6 + 1$. **24.** $1 + 4a^2 + 16a^4$. **25.** $a^6 + 2a^3y^3 + y^6$. **26.** $a^{2n} - 2a^nb^n + b^{2n}$. **27.** $x^{2n} + 2x^ny^n + y^{2n}$. **28.** $a^{2n} - b^{2n}$. **29.** $x^{2n} - y^{2m}$. **30.** $x^4 - bx^2 - cx^2 + bc$. **31.** $a^2 + 2ac + c^2 + ab + bc$. **32.** $m^3 + n^3$. **33.** $x^3 - y^3$. **34.** $a^3 - 3a^2b + 3ab^2 - b^3$. **35.** $x^4 - 2x^2y^2 + y^4$. **36.** $9a^2 + 30ab - 18ac + 25b^2 - 30bc + 9c^2$. **37.** $x^4 - y^4$. **38.** $m^8 - n^8$. **39.** $x^{16} - y^{16}$. **40.** $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$.

Page 38. — Ex. **3.** $x = 8$; $y = 1$. **4.** $x = 5$; $y = 4$. **5.** $x = 9$; $y = 4\frac{1}{3}$. **6.** $x = 4$; $y = 3$. **7.** $x = 8$; $y = 2$. **8.** $x = 1$; $y = 3$. **9.** $x = 1$; $y = 3$. **10.** $x = 7$; $y = 10$. **11.** $x = 7$; $y = 2$. **12.** $x = 6$; $y = 5$. **13.** $x = 1$; $y = 5$. **14.** $x = 10$; $y = 1$. **15.** $x = 8$; $y = 4$. **16.** $x = 4$; $y = 10$. **17.** $x = 3$; $y = 5$. **18.** $x = 3$; $y = 4$.

Page 39. — Ex. **1.** 10; 4. **2.** 275; 125. **3.** 31; 32. **4.** Horse, $200; wagon, $100. **5.** A, $6,000; B, $4,000. **6.** Sugar, 5 cts.; tea, 60 cts. **7.** Men, $2; boys, $1.25. **8.** Oranges, 3 cts.; bananas, 2 cts. **9.** Horse, $150; cow, $30. **10.** Larger, 600 acres; smaller, 300 acres. **11.** Men, $3; boys, $1. **12.** 18; 12.

Page 40. — Ex. **13.** Cloth, $2; silk, $3. **14.** Corn, 50 cts.; wheat, 80 cts.

Page 41. — Ex. **3.** 3. **4.** -3. **5.** -3. **6.** 3. **7.** a. **8.** a. **9.** x^3. **10.** x^3. **11.** a^6. **12.** n^6. **13.** $-a^4$. **14.** $-a^9$. **15.** a. **16.** x^3. **17.** n^5. **18.** a^{2n}. **19.** $-x^{2n}$. **20.** a^{5n}. **21.** x^{n-2}. **22.** x^{3-n}.

Page 42. — Ex. **23.** $2ax^3$. **24.** $-3ax^2$. **25.** $-4ay^2$. **26.** $-3a^5$. **27.** $4a^2$. **28.** $-5b$. **29.** $4a^3b$. **30.** $-4x^2$. **31.** $-4mn$. **32.** a^{m-n}. **33.** a^{m-n}. **34.** $-x^{n-m}$. **35.** $-x^{2n-2m}$. **36.** a^{3n-3m}. **37.** $4a - 2b$. **38.** $2m - 5mn^2$. **39.** $7y^3 - 4xy$. **40.** $-7x^3 + 5x^2 + 4x$. **41.** $4x - 2$. **42.** $-x - 2y - y^2$. **43.** $-m^2 - mn - n^2$. **44.** $8m + 5am^2$. **45.** $x^4 + x^3 + x^2 + x + 1$. **46.** $x^n - 2x^{2n} - 3x^{3n}$. **47.** $y^n - y^{n-4} - y^{n-5}$. **48.** $x^{2n-2} + x^{2n-1} - x^{3n-4}$. **49.** $-3 + 7x - 9x^2$. **50.** $-m^2 + m + 1$. **51.** $5y - 6x^{-1}y - 9 + 10xy^2$. **52.** $-2m^2n - 3n^2 + 4mn + 5n^7$. **53.** $a^2b - b^2c + ac^2 + 5abc$. **54.** $2(a + b) - 3(a + b)^2 - 6(a + b)^3$.

Page 44. — Ex. **5.** $x - y$. **6.** $x^2 - 2xy + y^2$. **7.** $a^2 + b^2$. **8.** $a^2 - ab + b^2$; $a^4 - a^3b + a^2b^2 - ab^3 + b^4$. **9.** $m - n$. **10.** $x^4 - x^2y^2 + y^4$. **11.** $3x - 1$. **12.** $x^4 + 2x^3 + x^2 - 4x - 11$. **13.** $3a + b$. **14.** $a^3 + 3a^2b + 3ab^2 + b^3$.

Page 45. — Ex. **15.** $3x + 2y + z$. **16.** $a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8$. **17.** $a^8 - a^4b^4 + b^8$. **18.** $m^3 - 2$. **19.** $x^4 - x^3 + x^2 - x + 1$. **20.** $3m^2 - 2n + 1$. **21.** $y^2 + x^2$. **22.** $x^2 + 2x + 4$; $y^4 - 3y^2 + 9$. **23.** $x^n + 2$; $x^3 - 7$. **24.** $x + 2x^2 - 5x^3$. **25.** $3x^2 + 2xy + 5y^2$.

Page 47. — Ex. **1.** $a + b$. **2.** $a - b$. **3.** $a^2 + ab + b^2$. **4.** $a^3 + a^2b + ab^2 + b^3$. **5.** $a^3 - a^2b + ab^2 - b^3$. **6.** $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$. **7.** $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$. **8.** $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$. **9.** $a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6$. **10.** $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 +$

$ab^6 + b^7$. **11.** $a^7 - a^6b + a^5b^2 - a^4b^3 + a^3b^4 - a^2b^5 + ab^6 - b^7$. **12.** $a^8 + a^7b + a^6b^2 + a^5b^3 + a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8$. **13.** $x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8$. **14.** $x^4 - x^2y^2 + y^4$. **15.** $x^6 - x^3y^3 + y^6$. **16.** $x^9 + x^8y + x^7y^2 + x^6y^3 + x^5y^4 + x^4y^5 + x^3y^6 + x^2y^7 + xy^8 + y^9$. **17.** $x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8$. **18.** $x^8 - x^4y^4 + y^8$. **19.** $x^{12} - x^{10}y^2 + x^8y^4 - x^6y^6 + x^4y^8 - x^2y^{10} + y^{12}$. **20.** $x^4 - x^2y^3 + y^6$. **21.** $x^8 - x^4y^2 + y^4$. **22.** $a^{12} - a^9b + a^6b^2 - a^3b^3 + b^4$. **23.** $m^{24} - m^{18}n + m^{12}n^2 - m^6n^3 + n^4$. **24.** $x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. **25.** $x^2 - 3x + 9$. **26.** $a^2 + 4a + 16$. **27.** $1 + n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + n^8$. **28.** $a^{16} - a^{12}b + a^8b^2 - a^4b^3 + b^4$. **29.** $x^5 + x^4 + x^3 + x^2 + x + 1$. **30.** $m^2 + 5m + 25$.——**1.** $-7a + 5b + 46c$. **2.** $43a^2 - 5ab + 44b^2$. **3.** $-(d+a)n^3 + (a+d)n^2 + 8mn$.

Page 48.—Ex. **4.** $4m^2 + 16mn - n^2$. **5.** $-a^3 + 26a^2b - 4ab^2 + 23b^3$. **6.** $-6m^5 + 8m^4 - 3m^3 + m^2 - 6m + 2$. **7.** $m^4 + 2m^2n^2 + n^4$. **8.** $x^4 + x^2y^2 + y^4$. **9.** $m^2 + 2mb + b^2 - a^2$. **10.** $2x^2 - 3xy + y^2$. **11.** $m^2 - m - 6$. **12.** $x^2 - xy + y^2$. **13.** $-a - b - c$. **14.** 0. **15.** $a - 3b + c + 1$. **16.** $a^8 - a^6y^3 + a^4y^6 - a^2y^9 + y^{12}$. **17.** $a^{16} - a^8b^9 + b^{18}$. **18.** $x^{12} + x^9y^2 + x^6y^4 + x^3y^6 + y^8$. **19.** $x = -1$; $y = 1$. **20.** $x = 7$; $y = 8$. **21.** $x = -5$; $y = 2$.

Page 49.—Ex. **1.** $a^2 + 2ax + x^2$. **2.** $a^2 + 2an + n^2$. **3.** $x^2 + 2xy + y^2$. **4.** $y^2 + 2yb + b^2$. **5.** $b^2 + 2bc + c^2$. **6.** $m^2 + 2mn + n^2$.

Page 50.—Ex. **7.** $l^2 + 2ln + n^2$. **8.** $a^2 + 2am + m^2$. **9.** $r^2 + 2rt + t^2$. **10.** $x^2 + 4x + 4$. **11.** $x^2 + 6x + 9$. **12.** $y^2 + 8y + 16$. **13.** $a^2 + 10a + 25$. **14.** $4 + 4x + x^2$. **15.** $49 + 14m + m^2$. **16.** $25 + 10n + n^2$. **17.** $36 + 12x + x^2$. **18.** $4a^2 + 12ab + 9b^2$. **19.** $25a^2 + 40ab + 16b^2$. **20.** $9x^2 + 18xy + 9y^2$. **21.** $4a^2 + 20ax + 25x^2$.——**1.** $a^2 - 2ab + b^2$. **2.** $a^2 - 2ax + x^2$. **3.** $x^2 - 2xy + y^2$. **4.** $y^2 - 2yb + b^2$. **5.** $n^2 - 2mn + m^2$. **6.** $b^2 - 2bc + c^2$. **7.** $l^2 - 2ln + n^2$. **8.** $a^2 - 2am + m^2$. **9.** $s^2 - 2st + t^2$. **10.** $x^2 - 4x + 4$. **11.** $y^2 - 6y + 9$. **12.** $a^2 - 8a + 16$. **13.** $4 - 4x + x^2$. **14.** $9 - 6m + m^2$. **15.** $16 - 8x + x^2$. **16.** $4a^2 - 8ab + 4b^2$. **17.** $9x^2 - 18xy + 9y^2$. **18.** $16x^2 - 40xy + 25y^2$. **19.** $4x^2 - 20xy + 25y^2$. **20.** $49x^2 - 42xy + 9y^2$. **21.** $9m^2 - 42mn + 49n^2$.

Page 51.—Ex. **1.** $a^2 - c^2$. **2.** $m^2 - a^2$. **3.** $x^2 - n^2$. **4.** $x^2 - y^2$. **5.** $a^2 - x^2$. **6.** $4a^2 - b^2$. **7.** $a^2 - 9b^2$. **8.** $16a^2 - 4b^2$. **9.** $9x^2 - 25y^2$. **10.** $25x^2 - 25y^2$. **11.** $9m^2 - 100n^2$. **12.** $A^2 - 4B^2$. **13.** $y^2 - 9z^2$. **14.** $25z^2 - 16y^2$. **15.** $a^2x^2 - b^2y^2$. **16.** $a^2b^2 - 4c^2$. **17.** $4a^2b^2 - 9c^2$. **18.** $16m^2n^2 - 9x^2y^2$.

Page 52.—Ex. **1.** $m^2 + 5m + 6$. **2.** $x^2 + 9x + 25$. **3.** $y^2 + 7y + 12$. **4.** $a^2 - 5a + 6$. **5.** $n^2 - 10n + 21$. **6.** $c^2 - 15c + 50$. **7.** $x^2 + 3x - 40$. **8.** $x^2 + 8x - 20$. **9.** $y^2 + 4y - 21$. **10.** $m^2 - 5m - 24$. **11.** $x^2 - 2x - 120$. **12.** $y^2 - 13y + 36$. **13.** $a^2 + (x+y)a + xy$. **14.** $c^2 + (d+f)c + df$. **15.** $x^2 + (y+z)x + yz$. **16.** $a^2 + (c-b)a - bc$. **17.** $m^2 + (x-y)m - xy$. **18.** $a^2 + (b-c)a - bc$.

Page 53.—Ex. **1.** $2 \times 2 \times 5\,aabb$. **2.** $2 \times 2 \times 2 \times 2\,mmmnnn$. **3.** $2 \times 3 \times 3 \times 3\,xxxxxyyyy$. **4.** $2 \times 2 \times 2 \times 2 \times 2 \times 2\,aabbbcccc$.

5. $3 \times 3 \times 3 \times 3\, aaabbb$. **6.** $2 \times 2 \times 3 \times 3 \times 3\, mmmmmmnnn$. **7.** $2 \times 2 \times 2 \times 3 \times 5\, xxxxy$. **8.** $2 \times 2 \times 2 \times 2 \times 3 \times 3\, aabbbbccc$. **9.** $2 \times 2 \times 2 \times 3 \times 3 \times 3\, mmaann$. **10.** $5 \times 5 \times 5\, aaabbb$. **11.** $3 \times 5 \times 7\, xxxxyyyyyy$. **12.** $13 \times 13\, mmmmmmaaaaaaa$.

Page 54. — Ex. **2.** $(m+n)(m-n)$. **3.** $(c+d)(c-d)$. **4.** $(x+y)(x-y)$. **5.** $(x+1)(x-1)$. **6.** $(1+m)(1-m)$. **7.** $(2m+3n)(2m-3n)$. **8.** $(4a+5b)(4a-5b)$. **9.** $(2x+7y)(2x-7y)$. **10.** $3\times3(2b+3m)(2b-3m)$. **11.** $(xy+ab)(xy-ab)$. **12.** $(m^2+n^2)(m+n)(m-n)$. **13.** $(m^4+n^4)(m^2+n^2)(m+n)(m-n)$. **14.** $(m^8+n^8)(m^4+n^4)(m^2+n^2)(m+n)(m-n)$. **15.** $(a^4+b^2)(a^2+b)(a^2-b)$. **16.** $(a^n+b^n)(a^n-b^n)$. —— **19.** $(a+b+c)(a+b-c)$. **20.** $(x+y+z)(x+y-z)$. **21.** $(m+n+p)(m+n-p)$. **22.** $(a-x+y)(a-x-y)$. **23.** $(l-n+m)(l-n-m)$. **24.** $(x+y+z)(x-y-z)$. **25.** $(a+b-c)(a-b+c)$. **26.** $(c+m+n)(c-m-n)$. **27.** $(x+a-y)(x-a+y)$. **28.** $(x+y+m+n)(x+y-m-n)$. **29.** $(a+b+c+d)(a+b-c-d)$. **30.** $(x-y+z-v)(x-y-z+v)$. **31.** $(m+p)(m-2n-p)$. **32.** $(x+y-1)(x-y+3)$. **33.** $(ax+by+cz-dv)(ax+by-cz+dv)$. **34.** $(m^2+n^2+a^2-b^2)(m^2+n^2-a^2+b^2)$.

Page 55. — Ex. **36.** $(x+y+z)(x+y-z)$. **37.** $(x-a+c)(x-a-c)$. **38.** $(y+x+z)(y-x-z)$. **39.** $(m-n+p)(m-n-p)$. **40.** $(a-b+y)(a-b-y)$. **41.** $(m+n-p)(m-n+p)$. **42.** $(a+x+y)(a-x-y)$. **43.** $(x+y+1)(x+y-1)$. —— **3.** $(a-x)(a^2+ax+x^2)$. **4.** $(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$. **5.** $(a-b)(a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6)$. **6.** $(a^2-b^3)(a^4+a^2b^3+b^6)$. **7.** $(x-y)(x^2+xy+y^2)$. **8.** $(x^3-y)(x^6+x^3y+y^2)$. **9.** $(x-y^2)(x^4+x^3y^2+x^2y^4+xy^6+y^8)$. **10.** $(m^2-y^3)(m^8+m^6y^3+m^4y^6+m^2y^9+y^{12})$. **11.** $(x^3-y^4)(x^6+x^3y^4+y^8)$. **12.** $(m-5)(m^2+5m+125)$. **13.** $(n+6)(n-6)$. **14.** $(a-b^6)(a^2+ab^6+b^{12})$. **15.** $(1-b)(1+b+b^2)$. **16.** $(2-b^2)(4+2b^2+b^4)$. **17.** $(a^2-b^3)(a^8+a^6b^3+a^4b^6+a^2b^9+b^{12})$.

Page 56. — Ex. **3.** $(m+n)(m^2-mn+n^2)$. **4.** $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$. **5.** $(m+b)(m^6-m^5b+m^4b^2-m^3b^3+m^2b^4-mb^5+b^6)$. **6.** $(x^2+b^2)(x^4-x^2b^2+b^4)$. **7.** $(n^2+m^2)(n^8-n^6m^2+n^4m^4-n^2m^6+m^8)$. **8.** $(x^4+y^4)(x^8-x^4y^4+y^8)$. **9.** $(a^2+y^2)(a^{12}-a^{10}y^2+a^8y^4-a^6y^6+a^4y^8-a^2y^{10}+y^{12})$. **10.** $(x^2+y)(x^8-x^6y+x^4y^2-x^2y^3+y^4)$. **11.** $(m^2+n^3)(m^4-m^2n^3+n^6)$. **12.** $(x+3)(x^2-3x+9)$. **13.** $(a^2+b^5)(a^4-a^2b^5+b^{10})$. **14.** $(n+1)(n^4-n^3+n^2-n+1)$. **15.** $(a+b^4)(a^4-a^3b^4+a^2b^8-ab^{12}+b^{16})$. **16.** $(m^2+n^4)(m^4-m^2n^4+n^8)$. **17.** $(y^2+z^5)(y^4-y^2z^5+z^{10})$.

Page 57. — Ex. **2.** $(m+n)^2$. **3.** $(x+y)^2$. **4.** $(a+x)^2$. **5.** $(p+m)^2$. **6.** $(a+m)^2$. **7.** $(2a+b)^2$. **8.** $(a+3b)^2$. **9.** $(2a+3b)^2$. **10.** $(4x+3y)^2$. **11.** $(m+8)^2$. —— **13.** $(m-n)^2$. **14.** $(c-d)^2$. **15.** $(y-x)^2$. **16.** $(a-d)^2$. **17.** $(p-q)^2$. **18.** $(3x-y)^2$. **19.** $(4a-1)^2$. **20.** $(5a-4b)^2$. **21.** $(10x-8b)^2$. **22.** $(a-13)^2$.

Page 58. — Ex. **24.** $(a+2)(a+6)$. **25.** $(a+1)(a+12)$. **26.** $(x+4)(x+5)$. **27.** $(m+2)(m+10)$. **28.** $(x+1)(x+20)$. **29.** $(m+4)(m+6)$. **30.** $(x+3)(x+8)$. **31.** $(y+2)(y+12)$.

32. $(z+1)(z+24)$. **33**. $(n+2)(n+41)$. **35**. $(x-2)(x-7)$. **36**. $(a-2)(a-3)$. **37**. $(m-2)(m-11)$. **38**. $(y-5)(y-6)$. **39**. $(z-5)(z-10)$. **40**. $(m-7)(m-13)$. **41**. $(x-7)(x-15)$. **42**. $(y-8)(y-15)$. **43**. $(a-7)(a-11)$. **44**. $(z-5)(z-17)$. **46**. $(x+8)(x-2)$. **47**. $(y+9)(y-3)$. **48**. $(z+10)(z-4)$. **49**. $(m+11)(m-5)$. **50**. $(n+12)(n-6)$. **51**. $(x+11)(x-4)$. **52**. $(y+12)(y-3)$. **53**. $(z+13)(z-5)$. **54**. $(m+16)(m-3)$. **55**. $(n+24)(n-4)$.

Page 59. — Ex. **57**. $(x-12)(x+4)$. **58**. $(y-10)(y+6)$. **59**. $(m-12)(m+7)$. **60**. $(n-20)(n+4)$. **61**. $(z-15)(z+9)$. **62**. $(s-12)(s+10)$. **63**. $(l-30)(l+8)$. **64**. $(x-14)(x+10)$. **65**. $(y-10)(y+9)$. **66**. $(z-13)(z+6)$. —— **2**. $a(a+b)^2$. **3**. $a(a+b+c)(a+b-c)$. **4**. $x^2(x+1)^2$. **5**. $y(y+1)^3$. **6**. $x(x+3)(x+4)$. **7**. $y(y-16)(y+4)$. **8**. $x(x-y)(x^2+xy+y^2)$. **9**. $y(y+x)(y^4-y^3x+y^2x^2-yx^3+x^4)$. **10**. $a(a^4+b^4)(a^2+b^2)(a+b)(a-b)$. **11**. $m(m^2+n^2)(m^4-m^2n^2+n^4)(m+n)(m^2-mn+n^2)(m-n)(m^2+mn+n^2)$.

Page 60. — Ex. **1**. $(x^2+y^2)(x+y)(x-y)$. **2**. $2(x-1)^2$. **3**. $(m+9)^2$. **4**. $(a+b-c)(a-b+c)$. **5**. $(m^2+n^2)(m^4-m^2n^2+n^4)$. **6**. $(x^2+y^2)(x^8-x^6y^2+x^4y^4-x^2y^6+y^8)$. **7**. $(6y+1)^2$. **8**. $(x^8+y^8)(x^4+y^4)(x^2+y^2)(x+y)(x-y)$. **9**. $(y-13)(y+10)$. **10**. $(a+11)(a-9)$. **11**. $(x+a+b)(x-a-b)$. **12**. $n(1+n)(1-n+n^2)(1-n)(1+n+n^2)$. **13**. $(m-15)(m+8)$. **14**. $2\times2\times x\times y$. **15**. $(m-16)^2$. **16**. $(m-n+p)(m-n-p)$. **17**. $(a^2-b^2+c^2-d^2)(a^2-b^2-c^2+d^2)$. **18**. $(a+9)(a-10)$. **19**. $(x+11)(x-12)$. **20**. $(x+y+1)^2$. **21**. $m^5(m+1)(m^6-m^5+m^4-m^3+m-1)$. **22**. $(z^2+10)^2$. **23**. $(x-1)^2(x+1)$. **24**. $(x+1)(x^2+1)$.

Page 61. — Ex. **3**. $3xyz$. **4**. $6m^2n^2$. **5**. $4m^2n^2y^3$. **6**. $7l^2m^4n^2$. **7**. $(x-y)$. **8**. $(x+y)$. **9**. $5xy^2z^2$. **10**. $5(x-y)$.

Page 62. — Ex. **11**. $10x^2y$. **12**. $x+y$. **13**. $x+3$. **14**. x^2+y^2. **15**. $x+2$. **16**. $x+3$. **17**. $x+9$. **18**. x^3-y^3. **19**. $x-y$. **20**. $a+b+c$. **21**. $a+b-c$. **22**. $m+n$. **23**. x^2+y^2. **24**. $m-4$. **25**. $b+c$. **26**. $x+4$. **27**. $n+c$. **28**. x^2-x+1. **29**. $m+3y$. **30**. $m-4$.

Page 64. — Ex. **3**. $24x^3y^2z$. **4**. $60a^3b^3c^3$. **5**. x^8-y^8. **6**. x^8-y^8. **7**. x^3-y^6. **8**. x^6-1. **9**. $4(9x^2-1)$. **10**. y^6-1. **11**. $y^3-9y^2+26y-24$. **12**. $x^4+9x^3+11x^2-81x-180$. **13**. $x^{12}-y^{12}$. **14**. $x^3-76x+240$. **15**. $[a^2-(b+c)^2][a^2-(b-c)^2]$. **16**. a^6-b^6. **17**. $x^4-2x^2y^2+y^4$. **18**. $x^{12}-y^{12}$. **19**. $6(1+x+x^2+x^3+x^4+x^5)$. **20**. $x^4-2x^2y^2+y^4$. **21**. $n^3-5n^2-64n+320$. **22**. $a^3-11a^2-64a+704$. **23**. $m^3+9m^2+26m+24$. **24**. x^4-1. **25**. $x^4-2x^2y^2+y^4$. **26**. $a^4+2a^3-19a^2-8a+60$. **27**. $(m+n+a)^2(m+n-a)$.

Page 66. — Ex. **3**. $\frac{2m^2}{3n}$. **4**. $\frac{c^3}{2ab^4}$. **5**. $\frac{11m}{14n}$. **6**. $\frac{3y^2}{4x^3z}$. **7**. $\frac{4p^2}{5m^5n^2}$. **8**. $\frac{5b}{6}$. **9**. $\frac{x-y}{2y-3x}$. **10**. $\frac{a+2}{a-2}$. **11**. $\frac{x+2}{2x-3}$.

12. $\dfrac{x-1}{x^2-x+1}$. **13.** $\dfrac{x+y}{x-y}$. **14.** $\dfrac{a+11}{a+12}$. **15.** $\dfrac{m-9}{m-10}$.
16. $\dfrac{a^2+ab+b^2}{a^4+a^3b+a^2b^2+ab^3+b^4}$. **17.** $\dfrac{a^2-ab+b^2}{(a+b)^2}$. **18.** $\dfrac{m+n-a}{m-n-a}$.
19. $\dfrac{x-y+z}{x+y+z}$.

Page 67. — Ex. **3.** $x^2+xy+y^2+\dfrac{2y^3}{x-y}$. **4.** $x^4-x^3y+x^2y^2-xy^3+y^4$.
5. $m^4-m^2n^2+n^4$. **6.** $x^2-xy+y^2-\dfrac{2y^3}{x+y}$. **7.** $a^2+ab+b^2+\dfrac{b^3}{a-b}$.
8. $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$. **9.** $x+a$. **10.** $x^2-y^2+\dfrac{2y^4}{x^2+y^2}$.
11. $1-\dfrac{3ab}{(a+b)^2}$. **12.** $x^2-2x+8-\dfrac{10}{x+2}$. **13.** $3x+2-\dfrac{3x-7}{4x^2+1}$.
14. $a^2-a+\dfrac{a-1}{a^2+a+1}$. **15.** $5x-14+\dfrac{63}{x+4}$. **16.** $m-n-\dfrac{3}{m+n}$.
17. $x^2+1+\dfrac{x+3}{x^2-x-1}$.

Page 68. — Ex. **2.** $\dfrac{2x}{x-y}$. **3.** $\dfrac{x^2-x}{x+1}$. **4.** $\dfrac{4ab}{a+b}$. **5.** $\dfrac{2ab}{a+b}$.
6. $\dfrac{2xy+2x^2}{x+y}$. **7.** $\dfrac{m^2+n^2}{m+n}$. **8.** $\dfrac{5x^2-2x-1}{2x-1}$. **9.** $\dfrac{x^2y^2}{x^2-xy+y^2}$.
10. $\dfrac{6a^2b}{a+b}$. **11.** $\dfrac{2}{x+1}$.

Page 70. — Ex. **2.** $\dfrac{a^2}{ab}$; $\dfrac{b^2}{ab}$. **3.** $\dfrac{my}{ny}$; $\dfrac{nx}{ny}$. **4.** $\dfrac{ac}{4c}$; $\dfrac{8}{4c}$. **5.** $\dfrac{2by}{x^2y^2}$; $\dfrac{3bx}{x^2y^2}$.
6. $\dfrac{3a}{4ab}$; $\dfrac{3b}{4ab}$. **7.** $\dfrac{2n}{5mn}$; $\dfrac{2m}{5mn}$. **8.** $\dfrac{amn}{m^2n^2}$; $\dfrac{bn^2}{m^2n^2}$; $\dfrac{cm}{m^2n^2}$. **9.** $\dfrac{3c^2}{12ac}$;
$\dfrac{3a}{12ac}$; $\dfrac{8n}{12ac}$. **10.** $\dfrac{2b}{abc}$; $\dfrac{5a}{abc}$; $\dfrac{4c}{abc}$. **11.** $\dfrac{a^2-ab}{a^2-b^2}$; $\dfrac{ab+b^2}{a^2-b^2}$; $\dfrac{c}{a^2-b^2}$.
12. $\dfrac{m}{a^3-b^3}$; $\dfrac{n(a^2+ab+b^2)}{a^3-b^3}$; $\dfrac{p(a-b)}{a^3-b^3}$. **13.** $\dfrac{a-1}{a^2-1}$; $\dfrac{a+1}{a^2-1}$; $\dfrac{1}{a^2-1}$.
14. $\dfrac{2x(x+y)}{x^3+y^3}$; $\dfrac{3y}{x^3+y^3}$; $\dfrac{4z(x^2-xy+y^2)}{x^3+y^3}$. **15.** $\dfrac{x+2}{x^3+3x^2-4x-12}$;
$\dfrac{2x-4}{x^3+3x^2-4x-12}$; $\dfrac{3x+9}{x^3+3x^2-4x-12}$. **16.** $\dfrac{a(x^2-xy+y^2)}{x^4+x^2y^2+y^4}$;
$\dfrac{b(x^2+xy+y^2)}{x^4+x^2y^2+y}$; $\dfrac{c}{x^4+x^2y^2+y^4}$.

Page 71. — Ex. **2.** $x=10$. **3.** $x=6$. **4.** $x=12$. **5.** $x=20$.
6. $x=24$. **7.** $x=12$. **8.** $x=6$. **9.** $x=12$.

Page 72. — Ex. **10.** $x=60$. **11.** $x=315$. **12.** $x=42$. **13.** $x=6$.
14. $x=15$. **15.** $x=-2\frac{1}{2}$. **16.** $x=12$. **17.** $x=-7\frac{1}{14}$. **18.** $x=60$.
19. $x=-1$ **20.** $x=35$. **21.** $x=6\frac{2}{7}$. **22.** $x=25$. **23.** $x=3$; $y=2$.
24. $x=9$; $y=8$. **25.** $x=60$; $y=48$. **26.** $x=6$; $y=8$. **27.** $x=2$;
$y=6\frac{2}{3}$. **28.** $x=37\frac{1}{2}$; $y=22\frac{1}{2}$.

Page 73.—Ex. **1.** 40. **2.** 60. **3.** 40; 20. **4.** A, \$4000; B, \$6000. **5.** \$8000. **6.** 49; 50. **7.** 21; 63. **8.** 20; 100. **9.** 30; 150. **10.** 120; 360.

Page 74.—Ex. **11.** 19; 61. **12.** 29; 121. **13.** \$900. **14.** 40 miles. **15.** 90 miles. **16.** 64 miles. **17.** $4\frac{4}{59}$ days. **18.** 12 hours.

Page 76.—Ex. **3.** $\frac{a^2+b^2+c^2}{abc}$. **4.** $\frac{m^2+n^2+p^2}{mnp}$. **5.** $\frac{a+b+c}{d}$. **6.** $\frac{yz+xz+xy}{xyz}$. **7.** $\frac{2m}{m^2-n^2}$. **8.** $\frac{a^2+b^2}{a^2-b^2}$. **9.** $\frac{2(x^2+y^2)}{x^2-y^2}$. **10.** $\frac{m+mn-n^2}{m^2-n^2}$. **11.** $\frac{2}{x^2+4x+3}$. **12.** $\frac{m^2+m+mn+3n}{(m+1)(m+2)(m+3)}$. **13.** $\frac{3}{1-x^2}$. **14.** $\frac{a^2-b^2}{abc}$. **15.** $\frac{x-z}{xyz}$. **16.** $\frac{m^2-2mn-n^2}{m^2-n^2}$. **17.** 4. **18.** $\frac{4ab}{a^2-b^2}$. **19.** $\frac{2y}{x^2-y^2}$.

Page 77.—Ex. **20.** $\frac{1}{x^2+5x+6}$. **21.** $\frac{1}{x^2-5x+6}$. **22.** $\frac{1}{x^2+5x+6}$. **23.** $\frac{2x+1}{x^2+x-6}$. **24.** $\frac{2}{x^2+10x+24}$. **25.** $\frac{2x}{x^2-y^2}$. **26.** $\frac{2y}{x^2-y^2}$. **27.** $\frac{2(x^2+1)}{x^4+x^2+1}$. **28.** $\frac{-4a}{a^4+a^2+1}$. **29.** $\frac{a^2+2ab^2+b^3}{a^3-b^3}$. **30.** $\frac{6x-27}{x^2-4}$. **31.** $\frac{x+2}{x^2-4x+3}$. **32.** $\frac{2}{x+1}$. **33.** $\frac{x^2-2x}{x^3+1}$.

Page 78.—Ex. **3.** $\frac{mx}{n}$. **4.** $\frac{a^2b}{cd}$. **5.** $\frac{6ab}{n}$. **6.** $\frac{ab^2}{c}$.

Page 79.—Ex. **7.** $\frac{2am}{n}$. **8.** $\frac{8bxy}{a}$. **9.** $\frac{4(a+y)}{a-y}$. **10.** $\frac{4a}{x^2+y^2}$. **11.** $\frac{4x}{x-1}$. **12.** $\frac{(a-y)(a+x)}{a-x}$. **13.** $\frac{2}{x^2-x+1}$. **14.** $\frac{3a(x^2+xy+y^2)}{x^2-2xy+y^2}$.

Page 80.—Ex. **3.** $\frac{a}{m}$. **4.** $\frac{2x}{3a}$. **5.** $\frac{a+x}{a-b}$. **6.** $\frac{1}{a^2-b^2}$. **7.** $\frac{m}{m+1}$. **8.** a. **9.** $\frac{a^2-ab+b^2}{a^2+ab+b^2}$. **10.** $\frac{m-n}{m+n}$. **11.** $\frac{x^2+4x-12}{x^2+8x+15}$. **12.** $a+b$. **13.** $\frac{a+b-c}{a-b-c}$. **14.** $\frac{a-y}{x+y}$. **15.** 1. **16.** 1. **17.** $\frac{a^2-ab+b^2}{a^6+b^6}$. **18.** $\frac{(m^2+n^2)(m+n)^2}{m^4+m^2n^2+n^4}$. **19.** 1.

Page 81.—Ex. **3.** $\frac{an}{bx}$. **4.** $\frac{n}{ax}$. **5.** $\frac{b}{mn}$. **6.** $\frac{2x}{5ab}$.

Page 82. — Ex. **7.** $\frac{7ax}{16m^2n^2}$. **8.** $\frac{18y}{5am^2n}$. **9.** $\frac{x^2+xy+y^2}{xy}$. **10.** $\frac{x+y}{x-y}$. **11.** $\frac{a-1}{a+1}$. **12.** $\frac{x-y}{xy}$. **13.** $\frac{a}{a-1}$. **14.** $\frac{x^2+x+1}{x+1}$. **15.** $\frac{y-3}{(y+5)^2}$.

Page 83. — Ex. **3.** $\frac{xy}{b}$. **4.** $\frac{2n}{3x}$. **5.** $\frac{20amz}{21cdy}$. **6.** $\frac{(m-n)^3}{(m+n)^2}$. **7.** $\frac{a^4-a^3b-ab^3+b^4}{a^4+a^3b+ab^3+b^4}$. **8.** $\frac{x-1}{x}$. **9.** $1-\frac{1}{x}$. **10.** $x+\frac{1}{x}$. **11.** $m^2+1+\frac{1}{m^2}$. **12.** $m^2-1+\frac{1}{m^2}$. **13.** $\frac{x^2+6x+9}{x^2-11x+18}$. **14.** $\frac{m^2+6m-16}{m^2+2m-15}$. **15.** 1.

Page 84. — Ex. **16.** $\frac{(a+b-c)^2}{(a-b-c)^2}$. **17.** $\frac{(m^2+n^2)^2}{m^4+n^4}$. **18.** $\frac{x^4}{2(x^2+y^2)}$.

Page 85. — Ex. **3.** $\frac{m-1}{m+1}$. **4.** $\frac{ac+b}{ac-b}$. **5.** $\frac{a^2+b^2}{a+b}$. **6.** 1. **7.** $\frac{b^2-an}{b^2+an}$. **8.** $\frac{am+2m+2}{am-2m-2}$. **9.** $\frac{a^2+a-2}{-a^2+a+2}$. **10.** $\frac{y+x}{y-x}$. **11.** $-\frac{x}{y}$. **12.** $\frac{x^2+1}{x^3+2x}$. **13.** $\frac{m^2-m-1}{m^3-m^2-2m+1}$. **14.** -1. **15.** $\frac{2ax}{a^2+x^2}$. **16.** $\frac{x^2-9x+20}{x^2+4x+3}$.

Page 87. — Ex. **2.** $x=3$; $y=2$. **3.** $x=4$; $y=5$. **4.** $x=6$; $y=4$. **5.** $x=6$; $y=4$. **6.** $x=4$; $y=2$. **7.** $x=7$; $y=2$. **8.** $x=6$; $y=9$. **9.** $x=6$; $y=8$. **10.** $x=10$; $y=20$. **11.** $x=12$; $y=8$. **12.** $x=7$; $y=3$. **13.** $x=12$; $y=5$. **14.** $x=5$; $y=2$. **15.** $x=2$; $y=3$. **16.** $x=5$; $y=10$. **17.** $x=3$; $y=2$. **18.** $x=12$; $y=35$. **19.** $x=-15$; $y=-30$.

Page 88. — Ex. **2.** $x=1$; $y=2$. **3.** $x=2$; $y=1$. **4.** $x=3$; $y=2$. **5.** $x=\frac{1}{2}$; $y=\frac{1}{3}$.

Page 89. — Ex. **6.** $x=3$; $y=4$. **7.** $x=5$; $y=10$. **8.** $x=1$; $y=6$. **9.** $x=4$; $y=2$. **10.** $x=7$; $y=8$. **11.** $x=6$; $y=12$. **12.** $x=16$; $y=15$. **13.** $x=20$; $y=10$. **14.** $x=5$; $y=4$. **15.** $x=2$; $y=11$. **16.** $x=3$; $y=12$. **17.** $x=10$; $y=100$.

Page 90. — Ex. **2.** $x=\frac{1}{2}(a+b)$; $y=\frac{1}{2}(a-b)$. **3.** $x=m$; $y=n$. **4.** $x=\frac{am-bn}{a^2-b^2}$; $y=\frac{an-bm}{a^2-b^2}$. **5.** $x=\frac{a+nb}{m+n}$; $y=\frac{a-mb}{m+n}$. **6.** $x=\frac{2mn}{m+n}$; $y=\frac{n^2-nm}{m+n}$. **7.** $x=\frac{m+n}{a}$; $y=\frac{m-n}{b}$. —— **1.** $x=20$; $y=27\frac{3}{4}$. **2.** $x=8$; $y=3$. **3.** $x=7$; $y=5$. **4.** $x=\frac{a^2bm-ab^2n}{a^2-b^2}$; $y=\frac{a^2bn-ab^2m}{a^2-b^2}$. **5.** $x=1$; $y=1\frac{1}{2}$. **6.** $x=-5$; $y=\frac{1}{3}$. **7.** $x=12$; $y=8$. **8.** $x=-7\frac{7}{9}$; $y=1\frac{1}{3}$.

Page 91. — Ex. 9. $x=2$; $y=3$. 10. $x=2$; $y=3$. 11. $x=\frac{2}{a+b}$; $y=\frac{2}{a-b}$. 12. $x=\frac{1}{2}$; $y=\frac{1}{2}$. 13. $x=\frac{1}{3}$; $y=\frac{1}{3}$. 14. $x=\frac{1}{4}$; $y=\frac{1}{3}$. 15. $x=\frac{m^2+n^2}{am+bn}$; $y=\frac{m^2+n^2}{an-bm}$. 16. $x=\frac{a^2-b^2}{2a-3b}$; $y=\frac{a^2-b^2}{3a-2b}$.

Page 92. — Ex. 2. $x=4$; $y=6$; $z=8$. 3. $x=2$; $y=3$; $z=1$. 4. $x=12$; $y=10$; $z=8$. 5. $x=3$; $y=5$; $z=7$. 6. $x=5$; $y=3$; $z=7$. 7. $x=12$; $y=24$; $z=36$. 8. $x=3$; $y=2$; $z=5$.

Page 93. — Ex. 9. $x=\frac{2}{3}$; $y=2$; $z=\frac{2}{3}$. 10. $x=5$; $y=6$; $z=7$. 11. $x=1\frac{1}{2}$; $y=\frac{1}{2}$; $z=\frac{1}{2}$. 12. $x=\frac{a+b}{2}$; $y=\frac{c-b}{2}$; $z=\frac{a-c}{2}$. 13. $x=10$; $y=20$; $z=30$. 14. $x=\frac{b+c}{2}$; $y=\frac{a-c}{2}$; $z=\frac{a-b}{2}$. 15. $x=\frac{2}{3}$; $y=2$; $z=1$. 16. $x=4$; $y=6$; $z=8$. 17. $x=a+b-c$; $y=a-b+c$; $z=-a+b+c$. 18. $x=\frac{a^2+ab+ac}{2}$; $y=\frac{ac-a^2+ab}{2}$; $z=\frac{ac-a^2-ab}{2}$. —— 1. $112\frac{1}{2}$ miles.

Page 94. — Ex. 2. 1969 men. 3. 3 miles. 4. 40 lbs. at 60 cts.; 80 lbs. at 90 cts. 5. Sugar, 5 cts.; tea, 60 cts. 6. Horses, $80; cows, $30. 7. Oranges, 3 cts.; bananas, 2 cts. 8. Lemons, 3 cts.; oranges, 4 cts.; bananas, 2 cts. 9. A, 8 rods; B, 6 rods. 10. A, $28\frac{4}{17}$ days; B, $20\frac{2}{23}$ days; C, $68\frac{4}{7}$ days.

Page 96. — Ex. 3. $9a^6x^4$. 4. $64x^6y^{12}$. 5. $-32a^{10}x^{10}$. 6. $-27m^6n^{12}$. 7. $a^8b^{12}c^4$. 8. $-125a^{18}b^{21}$. 9. $\pm a^{n^2}b^{n^2}$. 10. $a^{4n}b^{6n}$. 11. $64m^9n^{12}$. 12. $\pm a^{2n}m^{3n}$. 13. $a^{3n}b^6c^{3m}$. 14. $-32a^{5n}b^{5m}$. 15. $\frac{4a^4x^6}{25m^2n^4}$. 16. $\frac{m^{10}n^{15}}{x^{15}y^{20}}$. 17. $\frac{81a^8b^{12}}{625c^{12}d^4}$. 18. $\frac{64x^{3m}}{y^{3n}}$. 19. $\pm\frac{x^{2n}}{y^{3n}}$. 20. $-\frac{x^{12}n^8}{64m^8}$. 21. $\frac{m^{2n}x^n}{n^ny^{3n}}$. 22. $\frac{9x^4}{4y^6}$. 23. $\frac{x^8y^{12}}{16a^4b^8}$.

Page 97. — Ex. 2. $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$. 3. $x^2+y^2+z^2+u^2+2xy+2xz+2xu+2yz+2yu+2zu$. 4. $h^2+t^2+u^2+2ht+2hu+2tu$. 5. $x^2+y^2+z^2+u^2-2xy+2xz-2xu-2yz+2yu-2zu$. 6. $m^2+n^2+p^2+q^2+o^2+2mn-2mp+2mq-2mo-2np+2nq-2no-2pq+2po-2qo$. 7. $l^2+m^2+p^2+q^2+s^2-2lm+2lp-2lq+2ls-2mp+2mq-2ms-2pq+2ps-2qs$. 8. $a^2+l^2+g^2+e^2+b^2+r^2+2al+2ag+2ae+2ab+2ar+2lg+2le+2lb+2lr+2ge+2gb+2gr+2eb+2er+2br$. 9. $x^2+n^2+m^2+p^2+z^2-2xn-2xm-2xp+2xz+2nm+2np-2nz+2mp-2mz-2pz$.

Page 98. — Ex. 2. $m^2+(2m+n)n$. 3. $a^2+(2a+b)b$. 4. $t^2+(2t+u)u$. 5. $h^2+(2h+t)t+[2(h+t)+u]u$. 6. $m^2+(2m+n)n-[2(m+n)-p]p+[2(m+n-p)+q]q$. 7. $x^2-(2x-y)y+[2(x-y)+z]z-[2(x-y+z)-u]u$.

Page 99. — Ex. **2.** $x^3+3x^2y+3xy^2+y^3$. **3.** $m^3+3m^2x+3mx^2+x^3$. **4.** $8m^2+36m^2n+54mn^2+27n^3$. **5.** $a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3$. **6.** $a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3+3(a+b+c)^2d+3(a+b+c)d^2+d^3$. **7.** $h^3+3h^2t+3ht^2+t^3+3(h+t)^2u+3(h+t)u^2+u^3$. **8.** $x^3+3x^2y+3xy^2+y^3+3(x+y)^2z+3(x+y)z^2+z^3+3(x+y+z)^2u+3(x+y+z)u^2+u^3$. **9.** $l^3+3l^2m+3lm^2+m^3+3(l+m)^2n+3(l+m)n^2+3(l+m+n)^2o+3(l+m+n)o^2+o^3+3(l+m+n+o)^2p+3(l+m+n+o)p^2+p^3$. —— **1.** $t^3+(3t^2+3tu+u^2)u$. **2.** $h^3+(3h^2+3ht+t^2)t+[3(h+t)^2+3(h+t)u+u^2]u$. **3.** $T^3+[3T^2+3Th+h^2]h+[3(T+h+t)^2+3(T+h+t)u+u^2]u$. **4.** $a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c+[3(a+b+c)^2+3(a+b+c)d+d^2]d$. **5.** $m^3+(3m^2n+3mn+n^2)n+[3(m+n)^2+3(m+n)o+o^2]o+[3(m+n+o)^2+3(m+n+o)p+p^2]p$. **6.** $l^3+(3l^2+3la+a^2)a+[3(l+a)^2+3(l+a)n+n^2]n+[3(l+a+n)^2+3(l+a+n)d+d^2]d+[3(l+a+n+d)^2+3(l+a+n+d)s+s^2]s$.

Page 100. — Ex. **2.** $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. **3.** $a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$. **4.** $m^6+6m^5n+15m^4n^2+20m^3n^3+15m^2n^4+6mn^5+n^6$. **5.** $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$. **6.** $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$. **7.** $x^9-9x^8y+36x^7y^2-84x^6y^3+126x^5y^4-126x^4y^5+84x^3y^6-36x^2y^7+9xy^8-y^9$. **8.** $a^{10}-10a^9b+45a^8b^2-120a^7b^3+210a^6b^4-252a^5b^5+210a^4b^6-120a^3b^7+45a^2b^8-10ab^9+b^{10}$. **9.** $m^{11}-11m^{10}n+55m^9n^2-165m^8n^3+330m^7n^4-462m^6n^5+462m^5n^6-330m^4n^7+165m^3n^8-55m^2n^9+11mn^{10}-n^{11}$. **10.** $x^9+9x^8n+36x^7n^2+84x^6n^3+126x^5n^4+126x^4n^5+84x^3n^6+36x^2n^7+9xn^8+n^9$. **11.** $m^6-6m^5n+15m^4n^2-20m^3n^3+15m^2n^4-6mn^5+n^6$. **12.** $x^7-7x^6y+42x^5y^2-70x^4y^3+70x^3y^4-42x^2y^5+7xy^6-y^7$. **13.** $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.

Page 102. — Ex. **2.** $\pm 5ab^2$. **3.** $-4a^2b^3$. **4.** $\pm 7mn^2x^3$. **5.** $\pm 4ab^2c^3$. **6.** $-2mn^2x^3$. **7.** $a^2b^3c^5x^4$. **8.** $-8a^4n^3m^2x$. **9.** $\pm a^nb^{2m}c^{16}d^{1n}$. **10.** $\pm\dfrac{4xy^2}{6a^2b^3}$. **11.** $\dfrac{2mn^2}{3a^3b^4}$. **12.** $-\dfrac{ab^2}{2x^2y^3}$. **13.** $\pm\dfrac{a^2b^3}{5m}$.

Page 103. — Ex. **3.** $x+1$. **4.** $m+n$. **5.** $y+2$. **6.** $3x+2$. **7.** $x-y$. **8.** $4x-3y$. **9.** a^n-b^n. **10.** $x^{2n}+y^{2n}$.

Page 104. — Ex. **11.** $m^2+2mn+3n^2$. **12.** x^2-x-2. **13.** m^3-m^2-m-1. **14.** $4n^3-3n^2+2n-1$. **15.** a^3-2a^2+5a+3.

Page 106. — Ex. **3.** 65. **4.** 72. **5.** 85. **6.** 47. **7.** 238. **8.** 567. **9.** 876. **10.** 972. **11.** 2004. **12.** 3405. **13.** 3764. **14.** 15,813.

Page 109. — Ex. **4.** $m-n$. **5.** $1-2m$. **6.** y^2+y-1. **7.** x^3+3x-9. **8.** $a-b$. **9.** m^2-n^2. **10.** $1-3x+4x^2$. **11.** y^2-3y+2. **12.** a^2-2a+1. **13.** a^2-3a+2. **14.** y^3-y^2+y-1.

Page 113. — Ex. **3.** 63. **4.** 64. **5.** 85. **6.** 83. **7.** 76. **8.** 93. **9.** 123. **10.** 325. **11.** 345. **12.** 392. **13.** 652. **14.** 4321.

Page 115. — Ex. **3.** ± 8. **4.** ± 11. **5.** ± 12. **6.** ± 4. **7.** ± 7. **8.** $\pm\sqrt{a+b}$. **9.** ± 10. **10.** $\pm\sqrt{ac-ab}$. **11.** $\pm\sqrt{\frac{1+b}{c}}$. **12.** $\pm\sqrt{\frac{b+d}{a-c}}$. **13.** ± 5. **14.** $\pm\sqrt{\frac{p}{n-m}}$. **15.** ± 6. **16.** ± 3. **17.** ± 8. **18.** $\pm\sqrt{7}$. **19.** ± 7. **20.** ± 2.——**1.** ± 12. **2.** 20. **3.** Length, 160 rods; width, 32 rods. **4.** 27 men. **5.** ± 18.

Page 116. — Ex. **6.** 123. **7.** ± 10. **8.** ± 6. **9.** ± 8. **10.** ± 30. **11.** ± 65.

Page 118. — Ex. **4.** 4; -10. **5.** 15; -5. **6.** 21; -1. **7.** 6; -18. **8.** 3; -23. **9.** 12; -8. **10.** 10; -18. **11.** 1; -15. **12.** 18; -2. **13.** 20; -2. **14.** 3; -33. **15.** 25; -1. **16.** 21; -19. **17.** -2; -12.

Page 119. — Ex. **3.** 3; $-3\frac{1}{2}$. **4.** 3; $-2\frac{1}{3}$. **5.** 2; $-2\frac{1}{4}$. **6.** 5; $-3\frac{2}{3}$. **7.** 2; $-5\frac{1}{2}$. **8.** 4; -5. **9.** 2; $-1\frac{1}{2}$. **10.** 3; $-4\frac{2}{3}$. **11.** 4; $-2\frac{2}{5}$. **12.** 5; $-6\frac{1}{3}$. **13.** 9; $-7\frac{1}{3}$. **14.** 2; $-2\frac{5}{8}$. **15.** 2; $-4\frac{2}{3}$. **16.** $\frac{-b\pm\sqrt{4ac+b^2}}{2}$.

Page 120. — Ex. **2.** 6; 12. **3.** 13; 20. **4.** 3; 25. **5.** 6. **6.** \$20. **7.** \$90. **8.** \$30. **9.** 20 rd.; 60 rd. **10.** 20 ft.; 28 ft.

Page 122. — Ex. **3.** $x=6$; $y=2$. **4.** $x=6$ or 2; $y=2$ or 6. **5.** $x=6$ or 4; $y=4$ or 6. **6.** $x=10$; $y=2$. **7.** $x=2$ or 9; $y=9$ or 2. **8.** $x=7$ or 2; $y=2$ or 7. **9.** $x=5$; $y=3$. **10.** $x=10$ or 14; $y=14$ or 10. **11.** $x=20$; $y=4$. **12.** $x=3$ or 6; $y=6$ or 3. **14.** $x=4$; $y=3$. **15.** $x=1$; $y=2$. **16.** $x=1$; $y=1$. **17.** $x=4$; $y=1$.

Page 124. — Ex. **24.** $x=\pm 1$ or ± 5; $y=\pm 5$ or ± 1. **25.** $x=\pm 4$ or ± 2; $y=\pm 2$ or ± 4. **26.** $x=\pm 1$ or ± 3; $y=\pm 3$ or ± 1. **27.** $x=1$ or 4; $y=4$ or 1. **28.** $x=\pm 2$ or ± 3; $y=\pm 3$ or ± 2. **29.** $x=1$; $y=4$. **30.** $x=7$; $y=3$. **31.** $x=3$ or 2; $y=2$ or 3. **32.** $x=8$ or -3; $y=1$ or -10. **33.** $x=\pm 2$ or ± 10; $y=\pm 10$ or ± 2. **34.** $x=7$ or 2; $y=2$ or 7. **35.** $x=4$ or 3; $y=3$ or 4.——**1.** 12; 2. **2.** 3; 6. **3.** 6; 4. **4.** ± 3; ± 1. **5.** 10; 15. **6.** 7; 1.

Page 126. — Ex. **3.** 23. **4.** 60. **5.** 88. **6.** 122. **7.** 124. **8.** $2n$. **9.** $2n-1$. **10.** $3n-2$.

Page 128. — Ex. **3.** 100. **4.** 210. **5.** 420. **6.** 1404. **7.** 855. **8.** n^2. **9.** n^2+n. **10.** $\frac{an(n+1)}{2}$.——**1.** 156. **2.** 300. **3.** 57,900 ft. **4.** $257\frac{1}{3}$ ft. **5.** 3, 5, 7, …. **6.** 1, 3, 5, …. **7.** 2, 4, 6, …. **8.** 6, 12, 18, …. **9.** 3, 7, 11. **10.** 3, 8, 13.

Page 130. — Ex. **3.** 2187. **4.** 2048. **5.** 6561. **6.** 1,048,576. **7.** 3^{n-1}. **8.** 2^n. **9.** $3^{n-1}a^n$. **10.** $\pm 3^{n-1}$. **11.** $\frac{1}{2^{n-1}}$. **12.** $\frac{1}{3^{n-1}}$.

Page 131. — Ex. **3.** 29,524. **4.** 8190. **5.** 4095. **6.** 8191. **7.** 44,287.

Page 132. — Ex. **8.** -341. **9.** 2^n-1. **10.** $\frac{1}{2}(3^n-1)$. **11.** $\frac{2^n-1}{2^{n-1}}$. **12.** $\frac{1}{2}\left(\frac{3^n-1}{3^{n-1}}\right)$.

Page 133. — Ex. **1.** 2. **2.** $1\frac{1}{2}$. **3.** $1\frac{1}{3}$. **4.** $1\frac{1}{5}$. **5.** $1\frac{1}{5}$. **6.** $\frac{3}{4}$. **7.** $\frac{1}{3}$. **8.** $\frac{2}{3}$. **9.** $\frac{1}{4}$. **10.** $\frac{2}{7}$. —— **1.** 1, 5, 25, $\cdots$. **2.** 1, a^2, a^4, $\cdots$. **3.** 1, 2, 4, 8, $\cdots$. **4.** 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$. **5.** \$4095. **6.** \$10,485.75.

Page 134. — Ex. **1.** 6. **2.** 14. **3.** 156. **4.** 76. **5.** 1624. **6.** 116. **7.** $84 = 84$. **8.** $196 = 196$. **9.** $36 = 36$. **10.** $2744 = 2744$. **11.** $216 = 216$. **12.** $11,111 = 11,111$. **13.** 5. **14.** 11. **15.** $33x^2 - 38xy - 23y^2$.

Page 135. — Ex. **16.** $10\sqrt{a+b} + 15bc - a$. **17.** $m^2 - 6mn + n^2$. **18.** $8a^4 + 6a^3 + 2a$. **19.** $7n - 2x + b + y$. **20.** $x + 5a - 12b + 7d$. **21.** $a^{2m+2n+2} - b^{2m+2n-2}$. **22.** $m^7 + 1$. **23.** $a^{16} + a^8b^8 + b^{16}$. **24.** $a^2 - ab + b^2 - ac - bc + c^2$. **25.** $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4 - x^{n-6}y^5 + x^{n-7}y^6 -$ etc. **26.** $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + x^{n-5}y^4 + x^{n-6}y^5 + x^{n-7}y^6 +$ etc. **27.** $x = 5$. **28.** $x^6 + x^5y^2 + x^4y^4 + x^3y^6 + x^2y^8 + xy^{10} + y^{12}$. **29.** $m^8 - n^8$. **30.** $6m^2 - 16mn + n^2$. **31.** $x^4 - y^{2n}$. **32.** $x^{2n} + 2x^ny^n + y^{2n} - x^2y^2$. **33.** $6561x^8 - 256$. **34.** $a^4 + 6a^2x + 9x^2$.

Page 136. — Ex. **35.** $x^4 + 2x^2y^3 + y^6$; $4a^4 - 12a^2x^3 + 9x^6$. **36.** $a^6 + b^6$. **37.** $2b - a - 2c$. **38.** $6a - 9x - 7y$. **39.** $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$; $x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7$. **40.** $a^4 - 18a^2 + 81$. **41.** $a^{16} - a^8b^8 + b^{16}$. **42.** $4x^2 - 12xy + 9y^2$; $n^2x^2 + 2nmxy + m^2y^2$. **43.** $4a^2b^2 - 9c^2$. **44.** $y^2 + 4y - 21$; $a^2 + (b-c)a - bc$. **45.** $4xy$; $(x^2 + xy + y^2)(x^2 - xy + y^2)$. **46.** $(a^2 + b^2)(a^4 - a^2b^2 + b^4)(a^2 + ab + b^2)(a^2 - ab + b^2)(a+b)(a-b)$; $(a^4 + b^4)(a^8 - a^4b^4 + b^8)(a^2 + b^2)(a^4 - a^2b^2 + b^4)(a^2 + ab + b^2)(a^2 - ab + b^2)(a + b)(a - b)$. **47.** $(x - 30)(x + 4)$; $(x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$. **48.** $(x + 45)(x + 8)$; $(x - 45)(x + 8)$. **49.** $x - a - b - c$. **50.** $x^4 - x^2y^2 + y^4$. **51.** $a^4 - 10a^3 + 32a^2 - 38a + 15$. **52.** $x^4 - xy^3 + x^3y - y^4$. **53.** $a^4 - 52a^2b^2 - 2a^3b + 98ab^3 + 147b^4$.

Page 137. — Ex. **54.** $(m + n + x + y)(m + n - x - y)$. **55.** $(x + y + 1)(x + y - 1)$. **56.** $(m^2 + n^2 + a^2 - b^2)(m^2 + n^2 - a^2 + b^2)$. **57.** $(a - b + y)(a - b - y)$. **58.** $(1 + m + n)(1 - m - n)$. **59.** $(a - 3)(a - 20)$; $(x - 33)(x - 3)$. **60.** $(a + 13)(x - 7)$; $(x + 21)(x + 21)$. **61.** $(m - 6)(m - 4)$; $(x - 21)(x - 5)$. **62.** $a + 2$. **63.** $m^2 - m$. **64.** $x + 1$. **65.** $x^3 - x^2y - 2xy^2 - xz^2 + x^2z - y^3 - z^2y + 2xyz + y^2z - z^3$. **66.** $\frac{a^4 - a^2b^3 + b^6}{x^2 - b^3}$. **67.** $\frac{x + 15}{x - 14}$. **68.** $\frac{x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8}{x^4 - x^2y^2 + y^4}$. **69.** $\frac{a^3 - a + 2}{a + 2}$. **70.** $\frac{12(x - 2)}{12(x + 3)}$, $\frac{4(x + 4)}{12(x - 3)}$, $\frac{3(x - 5)}{12(x + 3)}$.

Page 138. — Ex. **71.** $\frac{z - x - y}{(y - z)(x - z)}$. **72.** $\frac{a + b}{(a - b)(c - b)}$. **73.** $\frac{x + y + z}{x - y - z}$. **74.** $\frac{2ab}{a^2 + b^2}$. **75.** $\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}$; $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$. **76.** $\left(m^2 + \frac{n^2}{x^2}\right)\left(m^4 - \frac{m^2n^2}{x^2} + \frac{n^4}{x^4}\right)$; $\left(\frac{x}{y} - n\right)\left(\frac{x^2}{y^2} + \frac{nx}{y} + n^2\right)$. **77.** $\left(m + \frac{1}{m}\right)^2$;

$\left(x+\frac{1}{x}\right)^2$. **78.** $\frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4b^2}$. **79.** 1. **80.** $\frac{(a^2+b^2)^2}{x^4+y^4}$. **81.** $\frac{a+3}{a-3}$. **82.** $\frac{a^2-9a+20}{a^2+4a+3}$. **83.** $\frac{a^2-bc}{a^2+bc}$.

Page 139. — Ex. **84.** $x=5$; $y=3$. **85.** $x=7\frac{4}{5}$; $y=2\frac{3}{5}$. **86.** $x=13$; $y=3$. **87.** $x=\frac{mn(an+bm)}{m^2+n^2}$; $y=\frac{mn(bn-am)}{m^2+n^2}$. **88.** $x=\frac{ac-bd}{a^2-b^2}$; $y=\frac{ad-bc}{a^2-b^2}$. **89.** $x=m+n$; $y=m+n$. **90.** $x=\frac{20}{97}(9m+7n)$; $y=\frac{10}{97}(m-10n)$. **91.** Length, 40 ft.; width, 27 ft. **92.** A, 108 da.; B, $56\frac{16}{19}$ da.; C, $43\frac{1}{5}$ da.; D, 1080 da. **93.** Apples, $\frac{an-bm}{a^2-b^2}$¢; oranges, $\frac{am-bn}{a^2-b^2}$¢.

Page 140. — Ex. **94.** A, 30; B, 18. **95.** A's, \$1000; B's, \$1200. **96.** 150 voters. **97.** Principal, $\frac{an-bm}{n-m}$ dollars; rate, $\frac{100(a-b)}{bm-an}$%. **98.** \$47. **99.** M, 18 mi.; N, $14\frac{1}{16}$ mi. **100.** $a^2+l^2+g^2+e^2+b^2+v^2+o^2+2al+2ag+2ae+2ab+2av+2ao+2lg+2le+2lb+2lv+2lo+2ge+2gb+2gv+2go+2eb+2ev+2eo+2bv+2bo+2vo$; $4a^2+9b^2+36c^2+d^2+12ab+24ac+4ad+36bc+6bd+12cd$. **101.** $r^2+e^2+p^2+u^2+b^2+l^2+i^2+c^2+2re+2rp+2ru+2rb+2rl+2ri+2rc+2ep+2eu+2eb+2el+2ei+2ec+2pu+2pb+2pl+2pi+2pc+2ub+2ul+2ui+2uc+2bl+2bi+2bc+2li+2lc+2ic$; $1+2x+3x^2+4x^3+3x^4+2x^5+x^6$. **102.** $l^2+m^2+p^2+q^2+r^2+s^2-2lm-2lp-2lq-2lr+2ls+2mp+2mq+2mr-2ms+2pq+2pr-2ps+2qr-2qs+2rs$; $A^2+2ABx+(B^2+2AC)x^2+2(AD+BC)x^3+(C^2+2BD)x^4+2CDx^5+D^2x^6$.

Page 141. — Ex. **103.** $m^3+3m^2x+3mx^2+x^3-3(m+x)^2y+3(m+x)y^2-y^3$; $a^3-3a^2b+3ab^2-b^3-3(a-b)^2c+3(a-b)c^2-c^3-3(a-b-c)^2d+3(a-b-c)d^2-d^3$. **104.** $m^3-3m^2n+3mn^2-n^3-3(m-n)^2p+3(m-n)p^2-p^3+3(m-n-p)^2y+3(m-n-p)y^2+y^3$; $x^3+3x^2y+3xy^2+y^3-3(x+y)^2z+3(x+y)z^2-z^3-3(x+y-z)^2w+3(x+y-z)w^2-w^3$. **105.** $a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+28a^2b^6-8ab^7+b^8$. **106.** $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. **107.** $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$. **108.** $16a^4+32a^3b+24a^2b^2+8ab^3+b^4$. **109.** $243x^5-810x^4y+1080x^3y^2-720x^2y^3+240xy^4-32y^5$. **110.** $16x^8-96x^6y^2+216x^4y^4-216x^2y^6+81y^8$. **111.** $3125a^5-18{,}750a^4b+45{,}000a^3b^2-54{,}000a^2b^3+32{,}400ab^4-7776b^5$.

112. $a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+$ etc.

113. $a^n-na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+$ etc.

114. $2a^3+3a^2-a-1$. **115.** $a-2b+3c-4d$. **116.** $a-3b+4x-2y$.

117. $3m^2-2m+5$. **118.** $a-1+\frac{2}{a}$. **119.** $a^{2n}-2a^n+3$. **120.** 1111.

121. 8905. **122.** ± 4. **123.** ± 6. **124.** ± 12.

Page 142. — **Ex. 125.** ± 7. **126.** ± 5. **127.** 17; -3. **128.** 30; -10. **129.** -1; -29. **130.** 1; 9. **131.** $-2 \pm \sqrt{5}$. **132.** 0; $-(6 + 2a)$. **133.** $-2a \pm \sqrt{4a^2 + 5a - b}$. **134.** 132; -12. **135.** 2; -102. **136.** 10; 50. **137.** $\frac{1}{2}(n \pm \sqrt{4m + n^2}$. **138.** -2; $-2n$. **139.** $2n$; $-2m$. **140.** $3a + b$; $b - a$. **141.** 3; $-11\frac{1}{3}$. **142.** 1.844+ ft. **143.** 4¢ per dozen. **144.** $x = \frac{1}{2}, \frac{1}{3}$; $y = \frac{1}{3}, \frac{1}{2}$. **145.** $x = 6, -1$; $y = 2, -12$. **146.** $x = 2$; $-1\frac{1}{2}$; $y = 3, -4$. **147.** $x = \pm 4, \pm 5$; $y = \pm 5, \pm 4$. **148.** $x = 5, 10$; $y = 4, 6$. **149.** $x = 4, 6\frac{6}{7}$; $y = 2, -\frac{6}{7}$.

Page 143. — **Ex. 150.** $x = \frac{1}{5}$; $y = 1$. **151.** $x = 2, 4\frac{6}{11}$; $y = 5, -\frac{1}{11}$. **152.** $x = 3, -5\frac{1}{3}$; $y = 2, 6\frac{1}{6}$. **153.** $x = 2, 7$; $y = 7, 2$. **154.** $x = 8, -3$; $y = 1, -10$. **155.** $x = \pm 1, \pm 2$; $y = \pm 2, \pm 1$. **156.** 5; 7. **157.** $\pm 2, \pm 8$. **158.** Length, 20 rd.; width, 10 rd. **159.** A's, \$800; B's, \$450. **160.** 3 yd.; 4 yd. **161.** 3; 4.

Page 144. — **Ex. 162.** $x = \dfrac{-d \pm \sqrt{8ds + (2a - d)^2}}{2}$.

163. $x = \dfrac{d - 2a \pm \sqrt{(2a - d)^2 + 8ds}}{2d}$. **164.** $x = \dfrac{d \pm \sqrt{(2l + d)^2 - 8ds}}{2}$.

165. $x = \dfrac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2}$. **166.** 3. **167.** 48.